Thermodynamic Aspects Of Black Holes

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Dedicated to The Strong Shoulders of Giants

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Abstract

The thermodynamic stability of black holes and the intricacies of their phase structure have been persistent subjects of inquiry since the inception of black hole thermodynamics. Distinguishing features such as negative heat capacity and imaginary thermal fluctuations set black holes apart from conventional thermodynamic entities. This thesis delves into the exploration of these distinctive properties in Anti-de Sitter (AdS) spacetime, focusing on black hole solutions in both Einstein's gravity and Gauss-Bonnet gravity with Born-Infeld electromagnetic non-linearity.

The first major section of the thesis begins with an examination of black hole solutions resulting from the coupling of Born-Infeld electromagnetism to Einstein's theory and Gauss-Bonnet gravity in a constant AdS background across general spacetime dimensions. These black holes exhibit a second-order phase transition between stable and unstable phases. Employing the Ehrenfest scheme and Ruppeiner's state space geometry technique, we conduct a detailed analysis of the order of phase transition. Our investigation extensively probes the dependence of phase transition properties on Born-Infeld and Gauss-Bonnet parameters.

In the second part of the thesis, we extend our exploration to Gauss-Bonnet and Born-Infeld black holes in an extended phase space, drawing on insights from gauge/gravity duality. Here, we derive a modified form of the first law, termed the mixed first law, by treating Newton's constant as a thermodynamic parameter alongside the cosmological constant in the bulk. Our analysis of free energy reveals that these black holes continue to exhibit behaviour similar to van der Waals fluids, a characteristic often associated with charged black holes in extended black hole thermodynamics. The central charge of the boundary theory now appear in the bulk first law. We observe a breakdown of the recently discovered universal behaviour of critical central

charge for both Gauss-Bonnet and Born-Infeld black holes. Our findings establish a more general result, indicating that this universal behaviour is a distinctive feature exclusive to Einstein's gravity in four-dimensional spacetime. We further delve into an extensive examination of the influence of Born-Infeld and Gauss-Bonnet parameters on the phase structure of these black holes.

This thesis is based on the following publications

- Phase transitions in Born-Infeld AdS black holes in D-dimensions (**N. Kumar**, S. Bhattacharyya, & S. Gangopadhyay)
  Gen Relativ Gravit 52, 20 (2020)
- Phase transitions in D-dimensional Gauss–Bonnet–Born–Infeld AdS black holes, (**N. Kumar**, S. Gangopadhyay)
  Gen Relativ Gravit 53, 35 (2021)
- Phase transition structure and breaking of universal nature of central charge criticality in a Born-Infeld AdS black hole, (**N. Kumar**, S. Sen, and S. Gangopadhyay)
  Phys. Rev. D 106, 026005 (2022)
- Breaking of the universal nature of the central charge criticality in AdS black holes in Gauss-Bonnet gravity, (**N. Kumar**, S. Sen, and S. Gangopadhyay)
  Phys. Rev. D 107, 046005 (2023)

List of Abbreviations	
AdS	Anti-de Sitter
SBH	Schwarzschild Black Hole
RN BH	Reissner-Nordstrom Black Hole
RN-AdS BH	Reissner-Nordstrom-anti-de Sitter Black Hole
AdS/CFT	Anti-de Sitter/Conformal Field Theory
QCD	Quantum Chromodynamics
ADM	Arnowitt Deser Misner

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CHAPTER 1

Introduction

Black holes have been the most fascinating and extensively studied objects since their discovery in 1916 [1]. These are the singular solutions to Einstein's field equations, which describe the structure of spacetime everywhere except at the singularity. The history of these solutions is as captivating as their wizard properties. Theoretical research in the first part of the last century was primarily focused on their astrophysical significance. The study of compact objects by Chandrasekhar [2], and the gravitational collapse of a neutron star to a singularity by Oppenheimer and Snyder [3] emphasized the physical existence of these mysterious entities. The latter half of the century ushered in significant conceptual advancements in the field. Understandings of the causal properties of lorentzian spacetime [4–8] and singularity theorems [9–13], proved the robustness of these solutions. The Penrose process of energy extraction from a black hole [14, 15], and Hawking's area theorem [16] laid the foundations of the field of black hole mechanics [17].

The existence of black holes, in synergy with well-established physics, has been a conundrum that led to major revolutions in fundamental physics. One such puzzle arose from the violation of the second law of thermodynamics in the presence of a black hole. In an important step towards resolution, Bekenstein proposed the generalised second law of thermodynamics and associated black holes with *entropy* propotional to the area of the event horizon [18, 19]. Hawking considered quantum fields in the black hole spacetime and discovered that an asymptotic observer perceives thermal radiation at late time [20,21], known as the *Hawking radiation*. Subsequently, it was also demonstrated that the generalised second law remains valid even in the presence of Hawking radiation [22, 23]. These insights reshaped the understanding of the geometric variables, as the surface gravity and horizon area in the first law of black hole mechanics were replaced by temperature and entropy, respectively. Consequently, black holes were realised as thermal entities.

Our current understanding of black holes stands at the intersection of two highly successful theories: quantum mechanics and general relativity. The thermodynamic properties of black holes are believed to be the averaged quantities derived from the underlying microscopic theory rooted in the quantum nature of gravity. Despite the absence of a universally accepted quantum gravity theory, the field of black hole thermodynamics serves as a critical benchmark that any microscopic theory must adhere to. One of the most remarkable and counterintuitive results is the relationship between a black hole's entropy and its surface area. Any microscopic degrees of freedom that emerge from an appropriate theory must reproduce this area-dependent entropy. Another intriguing aspect of black holes is their behaviour regarding thermodynamic stability and phase transitions. Even though Schwarzschild black holes are stable classically [24, 25], they exhibit instability when subjected to thermal fluctuations in a semiclassical context. The challenges posed by negative heat capacity and infinite discontinuities in heat capacity are overlooked, even within the domain of thermodynamics. Certainly, obtaining a microscopic understanding of these phenomena presents an even greater challenge.
Within a restricted domain, relatively well-accepted approaches to quantum gravity; string theory [26, 27] and loop quantum gravity [28, 29], have successfully reproduced the Bekenstein-Hawking entropy by counting microstates. The low energy limit of string theory [30–32], has also hinted at potential modifications to the Einstein-Hilbert action. Additionally, various viable modifications have been pursued in a bottom-up approach [33, 34], aiming to bridge theoretical gaps. Not only from the standpoint of quantum gravity, but also at the classical level, the motivation to explore modified theories of gravity arises from multiple challenges associated with the cosmological observations [35, 36]. The extensive and intricate nature of the theory space falls beyond the scope of this thesis for comprehensive review.

This thesis focuses on addressing the thermal stability and phase transition properties of black holes through a bottom-up approach. To explore these issues, we considered two well-motivated extensions of the Einstein-Hilbert action: Born-Infeld electromagnetic fields coupled to general relativity and Gauss-Bonnet gravity. These extensions were chosen for their simplicity, each involving only one free parameter to be determined. The extension to Born-Infeld electromagnetism [37] serves to resolve the issue of infinities associated with self-energy by imposing a cutoff on the field strength at the classical level in electromagnetism. On the other hand, the inclusion of the Gauss-Bonnet term in gravity, which represents the first term beyond the Einstein-Hilbert term and the cosmological constant in the broader Lovelock gravity framework [38], ensures the avoidance of the Ostrogradski instability.

Throughout this research, we aim to explore both the traditional and innovative perspectives on black hole phase transitions within the framework of these two extensions. This chapter provides an in-depth examination of black hole thermodynamics and phase transition properties. Chapter two delves into the intricacies of thermal stability and the phase structure of Born-Infeld and Gauss-Bonnet black holes within a constant anti-de Sitter (AdS) spacetime. Moving forward, Chapter 3 introduces an exploration of the relatively new concept of extended black hole thermodynamics for the aforementioned black hole models. We discuss results and embark upon prospective future projects in Chapter 4.

1.1 A Brief Review of Black Hole Thermodynamics in Seventies

Stationary black hole solutions are special in general relativity, as they represent systems in thermal equilibrium, a concept that will be clarified below. These solutions possess a Killing vector that becomes null on a null hypersurface, which is referred to as the Killing Horizon. In the context of Schwarzschild geometry, this horizon concides with the event horizon for the time-like Killing vector. In the case of a Kerr black hole, it is associated with a linear combination of the time-like Killing vector and the rotational Killing vector. Given that the Killing vector follows null geodesics on the Killing horizon, the geodesic equation for the Killing vector K^μ can be expressed as

$$K^\mu \nabla_\mu K^\nu = \kappa K^\nu \ . \tag{1.1}$$

Here, κ represents a constant known as the *surface gravity*, which is not unique and depends on the parameterization. The value can be fixed by choosing a normalisation condition for the Killing vector. One such choise is $K_\mu K^\mu = -1$ at asymptotic infinity, which coicides with the four velocity of a static observer parameterised with proper time. This fixes the Killing vector globally, and uniquely determines the value of the surface gravity. Therefore, surface gravity can be interpreted as the acceleration of a test particle near the horizon (at the horizon), as measured by a static observer at asymptotic infinity. It can be shown that the surface gravity is constant over the event horizon [39, 40]. This is also know as the *zeroth law* of black hole mechanics.

If a small amount of mass is added to a black hole, its area increases, which, in general, can be associated to change in the local energy-momentum tensor. Consequently, there is a change in the asymptotic values of the black hole charges. The variation in asymptotic values of these charges to the first order is connected by a relation, which is calculated in [41]. The relation is known as the *first law* of black hole mechanics. The form of the first law in Planck units is

$$\delta M = \frac{\kappa}{8\pi}\delta A \ . \tag{1.2}$$

For the most general black hole in flat spacetime, the first law takes the form

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Phi\delta Q + \Omega\delta J \tag{1.3}$$

where Q is the electric charge and J is the angular momentum. The conjugate variables, Φ and Ω, represent the electric potential and the angular velocity, respectively. Using Raychaudhuri equation, which governs the rate of expansion of geodesics, along with energy conditions, Hawking [16] showed that the area of the event horizon always increases. This result is known as the *second law* of black hole mechanics. For any physical process, the form of the area law is given by

$$\delta A \geq 0 \ . \tag{1.4}$$

Also, based on the physical arguments concerning the interpretation of surface gravity for an asymptotic observer, it is reasonable to constrain its values to be non-negative. Negative values of surface gravity would imply repulsive gravity to asymptotic observer, which is counter intuitive. This principle is referred to as the *third law* of black hole mechanics. The expression of this law is

$$\kappa \geq 0\,. \tag{1.5}$$

These four laws are analogous to the laws of thermodynamics. At the classical level, the analogy cannot be stretch further as nothing comes out of the horizon. The existence of black holes (and more generally, the existence of a horizon) violates the second law of thermodynamics. Bekenstein proposed a possible solution by suggesting that the black holes have finite entropy proportional to the area of the event horizon. The proposal is based on heuristic arguments derived from quantum mechanics, as quantum mechanics imposes a lower bound on the energy of the particles that can exist inside the horizon. This approach resolves the issue of infinite entropy, assigning a black hole entropy equal to area of the event horizon up to a proportionality constant. The proposed entropy formula [18] of a black hole in standard units is

$$S = \eta k \frac{A}{l_p^2} \tag{1.6}$$

where k is Boltzmann constant, l_p is the Plank length and η is the proportionality constant. Bekenstein also proposed the *generalised second law* of thermodynamics. According to this proposal, for a bounded region containing a black hole, the total change in entropy of the black hole and the entropy of the region outside the black hole is always positive.
i.e.,

$$\Delta S_{BH} + \Delta S_{outside} \geq 0\,. \tag{1.7}$$

Classically, the possibility of violating this law exists, and it only holds when quantum mechanical reasoning is taken into account. This strongly suggests that a black hole should have an associated temperature, originating from the quantum nature of gravity. Despite the limitations of our understanding of quantum gravity, Hawking made significant progress in comprehending the thermal properties of black holes and discovered that black holes radiate [20]. Hawking considered quantum fields in curved spacetime geometry of a non-rotating black hole and discovered that the vacuum state from past null infinity ($\mathcal{I}^{-}$) evolves into a thermal bath with a Planckian spectrum at future null infinity ($\mathcal{I}^{+}$), with a temperature given by

$$T = \frac{\kappa}{2\pi} \,. \tag{1.8}$$

This temperature is known as the *Hawking temperature*. It also fixes the proportionality constant (η) to be $1/4$ when considering the first law. Consequently, a black hole behaves like a blackbody and radiates at the Hawking temperature.

L. Smarr discovered an analog of the Eular relation in black hole mechanics using the dimensional dependence of intensive and extensive parameters [42]. We have explicitly calculated the Smarr relation for a variety of black holes in this thesis. For a simple example of Schwarzschild black hole, the formula takes the form

$$M = 2TS \,. \tag{1.9}$$

With all these discoveries, the thermodynamic nature of black holes was established in the first half of the seventies of the last century. We will now delve into the stability and phase transition properties of black holes, discussed in the next sections.

1.2 Second Order Phase Transition

After successfully defining the thermodynamic variables that characterize black holes as thermodynamic systems, the next step in understanding them is to analyze their macroscopic behavior under changes in these variables. The thermal response function provides insights into how a system responds to temperature adjustments. This is where black holes diverge from conventional thermodynamic systems. The heat capacity of a Schwarzschild black hole in asymptotically flat spacetime is negative, a departure from the behavior of systems in standard thermodynamics. Thus, they are not stable under thermal fluctuations when placed in a thermal bath.

In 1977, Davies [43, 44] investigated the heat capacity and phase transition behavior of Kerr-Newman black holes in asymptotically flat spacetime, solely utilizing thermodynamic variables. Davies calculated the heat capacity ($C_{J,Q}$) of the black hole while keeping other charges constant (in this case, angular momentum J and charge Q). His analysis revealed an infinite discontinuity, a clear indicator of a phase transition. Since all first-order derivatives of the free energy, as represented by the angular frequency (Ω), electric potential Φ, and entropy S, remain continuous and finite at the phase transition point, this transition is classified as of second order. The heat capacity diagram shown in Figure (1.1), taken from [43], illustrates the situation. It is evident from the figure that small black holes exhibit negative heat capacity, while large black holes are stable. There exists a critical point at which a black hole undergoes a phase transition. The precise nature of this phase transition remains somewhat elusive, as no system in standard thermodynamics exhibits negative heat capacity. Furthermore, without knowledge of the microscopic degrees of freedom, it is challenging to discern the distinctions between these two states.

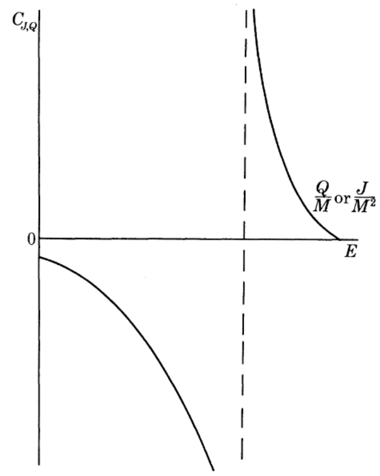


Figure 1.1: General behaviour of heat capacity at constant J and Q. For low values of J/M^2 and Q/M it is negative. The broken line indicates the position of the phase transition, at which $C_{J,Q}$ suffers an infinite discontinuity. Beyond this line it is positive, falling to zero at the extreme Kerr-Newman value E, which is the thermodynamic limit. **Proc. R. Soc. Lond. A 353: 499–521*

Expanding on this research, Chapter 2 will delve into the examination of the phase transition characteristics of Gauss-Bonnet black holes in asymptotically anti-de Sitter spacetime, taking into account the influence of Born-Infeld electromagnetic fields. Furthermore, we aim to en-

hance our comprehension of second-order phase transitions by employing diverse thermodynamic methodologies to characterize the nature of the phase transition.
Before delving further into the stability and phase transition properties of black holes, we will turn our attention to a quantum gravity formalism known as Euclidean quantum gravity, which has significantly enhanced our understanding of the thermodynamic properties of black holes. This exploration will serve as a fundamental basis for our subsequent discussions.

1.3 Euclidean Quantum Gravity and Black Hole Thermodynamics

One of the key advantages of Feymann's path integral approach in quantum theory apart from its conceptual appeal and the ease to deal field degrees of freedom, is related to its connection to statistical physics [45]. The imaginary time path integral describes a statistical system at a temperature identified by the periodicity of the imaginary time, also known as the Euclidean time. Buidling upon this, Gibbons and Hawking [46,47] extended this formalism to gravity and proposed that the functional integral of gravitational fields over an Euclidean section should be interpreted as the partition function of the canonical/grand-canonical ensemble. The path integral of quantum gravity for purely gravitational sector can be written as

$$Z = \int \mathcal{D}g e^{\iota S_L[g]} \tag{1.10}$$

Here, S_L is the gravitational action. When expressed in Euclidean form, i.e., under the transformation $t \to -\iota t$, the aforementioned partition function assumes the following structure:

$$Z = \int \mathcal{D}g e^{-S_E[g]}, \tag{1.11}$$

where, $S_E = -\iota S_L$ also known as Euclidean action. The new variable, imaginary time (t_E), can be demonstrated to possess a periodicity equal to the inverse of the Hawking temperature for a black hole spacetime. The intricacies associated with the challenges of computing these integrals fall outside the scope of this work. However, in the semiclassical approximation, where the primary contribution arises from the classical solution, the approximate form of the partition function is given by

$$Z \approx e^{-\underline{S_E}} \,. \tag{1.12}$$

Here, $\underline{S_E}$ is the on-shell action which is calculated using the classical solution. Now, the above partition function allows for the computation of various thermodynamic quantities. The free energy can be determined as follows

$$F = -T_H \ln Z. \tag{1.13}$$

Gibbons and Hawking employed this formalism to reproduce the Bekenstein-Hawking entropy formula. Not only did it demonstrate its consistency with thermodynamic quantities computed using other methods, but it also opened avenues to explore beyond the realm of thermodynamics. The complex calculations beyond the saddle-point approximation in the context of supersymmetry have also been utilized to compute corrections to the entropy formula, leading to logarithmic corrections [48–50]. Moreover, this formalism has provided crucial insights into the primary focus of this thesis: the physics of black hole phase transitions. Subsequent sections will highlight some of these significant insights.
In the following section, our focus will be on exploring another intriguing phase transition in black holes, known as the Hawking-Page phase transition.

1.4 Stability and The Hawking-Page Phase Transition

As we know, an uncharged static black hole solution, which appears to be stable classically, is thermodynamically unstable on the entire range of black hole mass. This statement can be understood for the simple case of Schwarzschild black hole, where the mass is related to the Hawking temperature as

$$M = \frac{1}{8\pi T} \,. \tag{1.14}$$

Consequently, the heat capacity is negative throughout. Thermal fluctuations can drive the system toward complete evaporation or the encapsulation of all matter inside it. Therefore, a canonical ensemble cannot be defined for these black holes. York [51] and York and Brown [52] proposed a resolution to this issue by offering a way to realise the partition function calculated using the Euclidean path integral approach. They demonstrated that confining an asymptotically flat black hole in a finite-sized box renders it stable. Page and Hawking [53] have previously demonstrated the thermal stability of the Schwarzschild black hole without any box/cavity in AdS spacetime. These stable black holes in AdS spacetime also undergo a phase transition also known as the Hawking-Page phase transition. The phase transition can be understood in the following way:

The Schwarzschild-AdS black hole metric in four spacetime dimensions is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{1.15}$$

where the lapse function $f(r)$ takes the form

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} \ . \tag{1.16}$$

The parameter l represents the AdS radius, which is related to the cosmological constant as $\Lambda = -\frac{3}{l^2}$. Also, the Hawking temperature of the black hole, expressed in terms of horizon radius (r_+), is given by

$$T = \frac{1}{4\pi} f'(r)\bigg|_{r_+} = \frac{1}{4\pi}\left(\frac{1}{r_+} + \frac{3r_+}{l^2}\right) \ . \tag{1.17}$$

It is plotted with the horizon radius for a scale set by the AdS radius $l = 1$. From Fig.(1.2), there exists a minimum temperture $\left(T_0 = \frac{\sqrt{3}}{2\pi l}\right)$ below which no black hole exists, and above it, two black hole branches exist with small and large masses. The small black hole is thermodynamically unstable, similar to the Schwarzschild black hole in flat spacetime, while the large black hole is stable. Below the temperature T_0, the reference space, thermal AdS exists, which can be realised as a state constructed by imposing a periodicity of arbitrary value to the Euclidean time as AdS spacetime itself is not periodic. One can also utilise the Euclidean ac-

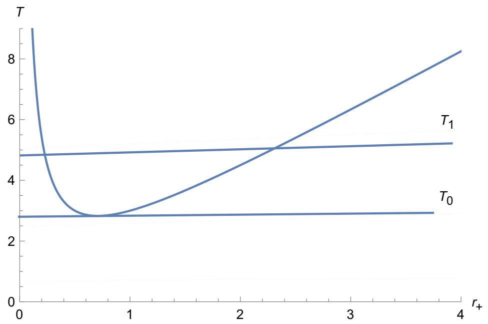


Figure 1.2: Hawking Temperature vs Horizon Radius for ($l = 1$)

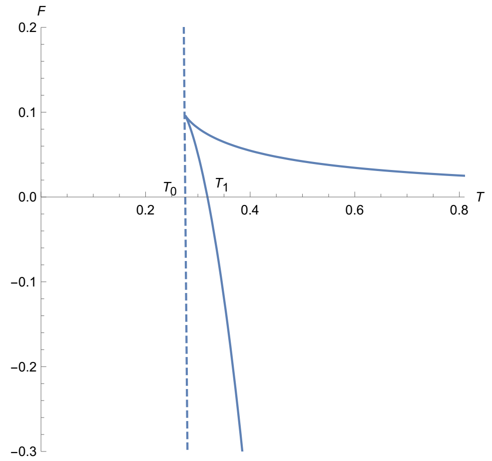


Figure 1.3: Free Energy (Free Energy Change) vs Hawking Temperature for ($l = 1$)

tions to calculate the free energy change (free energy difference) of Schwarzschild-AdS and thermal AdS. Qualitatively, this is also equivalent to calculating Helmoholtz free energy from the known thermodynamic variables or from the on-shell action by the counterterm subtraction method [46] (which involves subtracting Gibbons-Hawking term and a specific counter term). The background free energy of Hawking radiation (thermal AdS) will be a small negative value in comparison to the black hole free energy [53], which can be considered to get contribution from the metric fluctuations and matter fields. The free energy of thermal radiation is given by [53]

$$F_{thermal} = -\frac{\pi^2}{30} g l^3 T^4 \tag{1.18}$$

for some constant g. Using Bekenstein's entropy formula, we can calculate the free energy of the black hole as

$$F_{BH} = M - TS = \frac{r_+(l^2 - r_+^2)}{4} \ . \tag{1.19}$$

Fig.(1.3) confirms that only thermal AdS exists before T_0. Between temperature T_0 and T_1, thermal AdS is favoured over black hole solutions because the free energy change will be negative. It is evident that beyond temperature $\left(T_1 = \dfrac{1}{\pi l^2}\right)$, Schwarzschild-AdS is thermodynamically more stable. The point T_1 is also known as the Hawking-Page phase transition point. Beyond

this temperature, a more stable large black hole is prefered over thermal AdS spacetime or a small black hole. This condensation of radiation to a black hole is referred to as Hawking-Page phase transition, characterized as a first-order phase transition due to the abrupt change in entropy at T_1. This is also visible from the free energy plot, where the complete system (consisting of the black hole and thermal AdS) exhibits a kink in the free energy at T_1.
The case of charged black hole is particularly intriguing, as we can define both canonical and grand-canonical ensemble for these. These exhibit an interesting critical behaviour and small-large black hole phase transition which shall be discussed in the subsequent section.

1.5 Small-Large Black Hole Phase Transition

Andrew Chamblin discovered an interesting phase transition that is isomorphic to a van der Waals fluid in the canonical ensemble for Reissner-Nordstrom (RN) black holes [54, 55]. The charged black hole solution in AdS spacetime is a solution of the action

$$I = -\frac{1}{16\pi}\int dx^4\sqrt{-g}\left(R - F^2 + \frac{6}{l^2}\right) . \tag{1.20}$$

A static black hole solution is of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{1.21}$$

with lapse function, $f(r)$ being

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2} . \tag{1.22}$$

Since, the black hole is now characterised by its mass and charge. The first law takes the form

$$dM = TdS + \Phi dQ . \tag{1.23}$$

Here, Φ is the electric potential. Now, the thermodynamic properties of the black hole can be studied either for constant potential or constant charge. The constant potential ensemble represents the grand-canonical ensemble, while the constant charge ensemble represents the canonical ensemble. Andrew pointed out that for grand-canonical ensemble, pure AdS (i.e., thermal AdS) serves as a background because, at constant potential, both charged and uncharged quanta can exist, contributing to the temperature. In case of constant charge ensemble, localised charge does not represent a solution of electromagnetic fields in AdS spacetime. In this scenario, the extremal black hole solution is considered as a background. The free energy in both cases can be calculated using the Euclidean path integral approach by suitably subtracting the background. Alternatively, the expression for the free energy can be derived by leveraging thermodynamic variables as

$$F_\Phi = M - TS - \Phi Q \,, \qquad\qquad F_Q = M - TS \,. \tag{1.24}$$

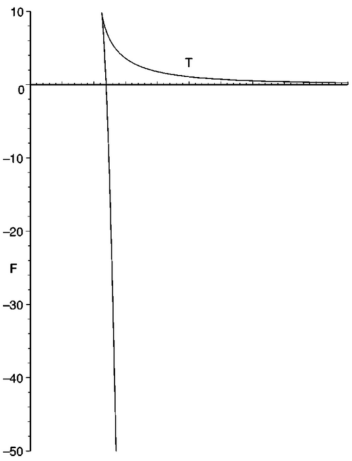


Figure 1.4: Free Energy (F_Φ) vs Temperature for $l = 1, \Phi = 10$ **Chamblin et. al. PRD, 60, 064018 (1999)*

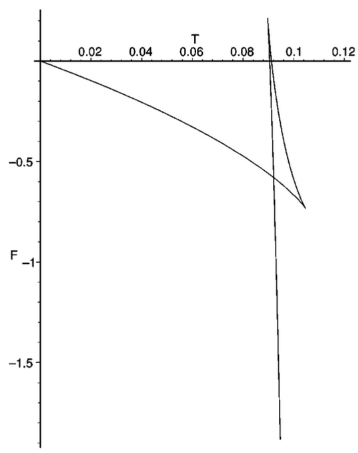


Figure 1.5: Free Energy (F_Q) vs Temperature for $l = 5,\ Q = 1.25$ **Chamblin et. al. PRD, 60, 064018 (1999)*

The Fig.(s)(1.4, 1.5) are taken from [54], where the free energy F_Φ illustrates the Hawking-Page phase transition in the grand-canonical ensemble, whereas F_Q, representing the canonical ensemble, exhibits a unique phase transition from a small to a large black hole as the temperature increases. The shallow-tail in the free energy diagram resembles a van der Waals fluid. Depending on the parameter choices, there exists a critical point at which the phase transition is of the second order, and below that, it is of the first order. We will discuss the small-large black hole phase transition in a context where the cosmological constant is considered as a thermodynamic variable in chapter 3 of this thesis.

The next section is dedicated to a brief survey of the Gauss-Bonnet gravity and black hole solutions in it. These black hole solution will be considered throughout this thesis.

1.6 Gauss-Bonnet Gravity and Black Hole Solution

It is widely acknowledged that General Relativity is not the final answer and must be succeeded by a theory that unifies it with the quantum framework at the small scale. Additionally, the presence of dark matter and dark energy in cosmology has been a puzzle for General Relativity. Numerous modifications proposed in the literature [35] dealing with these issues are undergoing rigorous testing through experimental observations. Recent detections of gravitational waves [56, 57] have propelled us beyond conventional electromagnetic observations, starting a new era for testing the strong gravity regime.

Our inclination to explore departures from General Relativity is primarily rooted in our pursuit of quantum gravity, driven by the desire to understand the phase transition properties of black holes. The inclusion of higher curvature invariants represents a common modification inspired by string theory [30, 31]. One such higher curvature theory is the Gauss-Bonnet gravity theory, which is part of a general extension known as the Lovelock theory [38]. The Gauss-Bonnet gravity theory does not involve higher-order time derivatives in the equation of motion. In four-dimensional spacetime, the Gauss-Bonnet term functions as a topological term; therefore, it does not contribute to the equation of motion. Thus, in four spacetime dimensions, the observations implicitly align with the general theory of relativity. Nevertheless, our focus extends to the study of higher-dimensional black holes, enabling us to examine the implications of this modification. For a comprehensive exploration of higher-dimensional black holes, Emparan and Reall [58] provided an extensive review, shedding light on the underlying motivations and the progress made in this area.

We start with the lagrangian density of the Lovelock theory. The first t terms of the theory in D spacetime dimensions are given by [59]

$$\mathcal{L} = \sqrt{-g}\sum_{n=0}^{t}\alpha_n\mathcal{R}^n, \qquad \mathcal{R}^n = \frac{1}{2^n}\delta^{\mu_1\nu_1\ldots\mu_r\nu_r}_{\alpha_1\beta_1\ldots\alpha_n\beta_n}\prod_{r=1}^{n}R^{\alpha_r\beta_r}{}_{\mu_r\nu_r}\,. \tag{1.25}$$

It is the most natural extension of Einstein's theory to higher dimensions which follows the

Lovelock theorem [38]. In explicit from, the lagrangian density with first three terms takes the form

$$\mathcal{L} = \sqrt{-g}[\alpha_0 + \alpha_1 R + \alpha_2(R^2 + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\mu\nu}R^{\mu\nu})] + \alpha_3\mathcal{O}(R^3)\ . \tag{1.26}$$

where α_0 is the cosmological constant, the second term is the Einstein term, and the third term is known as the Gauss-Bonnet term, with α_2 being the Gauss-Bonnet parameter (rather the normalised α_2/α_1 is called the Gauss-Bonnet parameter).

The black hole solutions in Gauss-Bonnet gravity were first discovered by Boulware and Deser [60], in both asymptotically flat and anti-de Sitter spacetime. A static, spherically symmetric solution in 5 spacetime dimensions takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \tag{1.27}$$

where the lapse function is of the form

$$f(r) = 1 + \frac{r^2}{4\alpha} + \sigma\frac{r^2}{4\alpha}\sqrt{1 + \frac{16\alpha M}{r^4} + \frac{4\alpha\Lambda}{3}}\ . \tag{1.28}$$

Here, $\alpha_0/\alpha_1 = -2\Lambda$ and $\alpha_2/\alpha_1 = \alpha$. The variable σ can have values of $+1$ or -1. It has been demonstrated that the solution with $\sigma = -1$ is well behaved, whereas $\sigma = 1$ exhibits ghost instabilities [60]. The charged black hole solution was discovered by Wiltshire [61]. Since then the themrodynamic properties of these black holes have been extensively studied in the literature [62–69], with this list far from complete. In this thesis, we aim to investigate the phase transition properties in both the general and extended thermodynamic phase space.

It is crucial to note that while these theories represent the most natural extension of Einstein's theory, they do not demonstrate entirely satisfactory causal behavior. Research has indicated that the Einstein-Gauss-Bonnet gravity theory allows for closed timelike curves, even when all energy conditions are satisfied [70]. Hence, it is vital to recognize that these theories function solely as effective field theories. Recently, Glavan and Lin [71] have also proposed a method to circumvent the constraints of the Lovelock theory by rescaling the Gauss-Bonnet parameter, introducing it into 4-dimensional black hole solutions. However, our focus has remained on exploring the conventional topological implications of the Gauss-Bonnet term, and we have not yet delved into this particular direction.

In the next section, we shall discuss the Born-Infeld electromagnetism and black hole solutions when gravity is coupled to these fields.

1.7 Born-Infeld Electromagnetism and Black Hole Solution

Born-Infeld electromagnetism [37] was formulated in 1934 to address the issue of divergences linked to point charges. Its development mirrored the approach of modifying Galilean mechanics to incorporate relativity, imposing a limit on the maximum value of the field strength and modifying the symmetry group. This mechanism shares similarities with the renormalization of electric charge with a length scale in quantum electrodynamics, but instead a cutoff on the electric field at small distances is enforced at classical level. The field strength and energy density of a point charge are illustrated in Fig. (1.6). Apart from the model's simplicity, the advantages

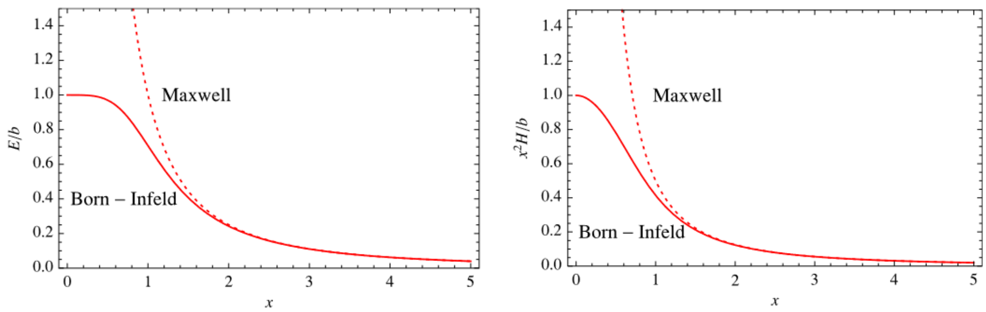


Figure 1.6: In this plot we show the regularisation occurring in Born–Infeld electromagnetism (solid lines) as compared to the case of Maxwell's theory (dotted lines). In the left panel we show the profile (as a function of $x \equiv \sqrt{4b/Qr}$) for the electric field generated by a point-like charge. We can clearly see the change from the usual $1/r^2$ behaviour at large distances to the saturation for the electric field due to the Born–Infeld corrections on small scales. In the right panel we show how this modified behaviour at small scales also regularises the energy density of the particle. **Physics Reports Volume 727, 11 January 2018, Pages 1-129*

of considering the Born-Infeld theory lie in its characteristics of electric-magnetic self-duality and its adherence to valid causal behavior. Moreover, it also appears in string theory as an appropriate option to couple gauge fields to open strings [72]. The Lagrangian density of the

Born-Infeld electromagnetism is given by

$$\mathcal{L} = 4b^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2b^2}}\right) \tag{1.29}$$

where b is the Born-Infeld parameter. Maxwell Lagrangian density is recovered in the limit $b \to \infty$. The most straightforward approach to integrating Born-Infeld electromagnetism with gravity is to couple it with General Relativity. While this integration may not resolve the issue of black hole singularities, it does yield intriguing black hole solutions. Extensive research on these black holes has been conducted in the literature [73–80] for various purposes. Of course, the list of references is far from exhaustive. The counterpart of Reissner-Nordstrom black hole in $3+1$ spacetime with cosmological constant takes the form [81]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{1.30}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{2b^2r^2}{3}\left(1 - \sqrt{1 + \frac{Q^2}{b^2r^4}}\right) + \frac{4Q^2}{3r^2}\ {}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{Q^2}{b^2r^4}\right]\ . \tag{1.31}$$

Here, ${}_2F_1[a,b,c,z]$ is the hypergeometric function. The solution reduces to the RN black hole in the limit $b \to \infty$. The thermodynamic properties of Born-Infeld black holes have also been extensively examined from various perspectives in the literature [82–91]. Our goal is to comprehend the impact of the parameter b on the thermodynamic stability properties.

Next, we will explore a relatively recent perspective on the thermodynamics of black holes known as Extended Black Hole Thermodynamics.

1.8 Extended Black Hole Thermodynamics

An insightful observation by Kastor et al. in 2009 [92] drew attention to a fundamental inconsistency between the first law and the Smarr relation for black holes in AdS spacetime. Notably, the mass relation derived from the first law did not align with the one obtained from the Smarr relation in asymptotic AdS spacetime. To reconcile this disparity, the authors proposed treating the AdS parameter as a thermodynamic variable, identifying Λ as the black hole's pressure, and incorporating a pressure-volume term in the first law. Additionally, they provided a geometric derivation of the Smarr relation. This expansion of the thermodynamic parameter space is known as extended black hole thermodynamics or black hole chemistry. In this context, we will briefly outline an argument based on the scaling of thermodynamic variables in support of the extension of the thermodynamic phase space, a concept also discussed in [92]. We will consider the case of a simple Schwarzschild black hole in AdS spacetime, with the metric given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \tag{1.32}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} \; . \tag{1.33}$$

The mass of the black hole in terms of the horizon radius is expressed through the relation $f(r+) = 0$, taking the form

$$M = \frac{r_+}{2} + \frac{r_+^3}{2l^2} \; . \tag{1.34}$$

The Hawking temperature and entropy of the black hole are given by

$$T = \frac{1}{4\pi}\left(\frac{1}{r_+} + \frac{3r_+}{l^2}\right), \qquad\qquad S = \pi r_+^2 \; . \tag{1.35}$$

The Smarr relation [42] serves as an alternate formula for the internal energy (mass of a black hole), expressed in terms of extensive and intensive variables, similar to Euler's relation in standard thermodynamics. It can be derivated using Euler's theorem for quasi-homogeneous functions and the first law of thermodynamics. In the case of the aforementioned black holes, we consider the mass as a function of entropy, i.e., $M \equiv M(S)$, under a constant cosmological background (with a fixed l). Referring to equations (1.34) and (1.35), it becomes evident that during an overall change in the length scale, the quantities M and S scale as $M \propto \lambda$ and $S \propto \lambda^2$.

Using Euler's theorem and the first law, the mass becomes

$$M = 2TS \; . \tag{1.36}$$

Euler's Theorem

According to Euler's theorem of quasi-homogeneous functions:
If

$$f(\lambda^a \, x, \lambda^b \, y) = \lambda^c \, f(x, y) \tag{1.37}$$

then

$$c \; f(x, y) = a \; x\frac{\partial f}{\partial x} + b \; y\frac{\partial f}{\partial y} \tag{1.38}$$

for some scaling parameter λ.

Euler's Relation in Thermodynamics

First law of thermodynamics is given by

$$dE = TdS - PdV + \mu dN \; . \tag{1.39}$$

Now, consider internal energy $E \equiv E(S, V, N)$. We know that under overall change of scale of the system by λ, we have

$$E(\lambda S, \lambda V, \lambda N) = \lambda E(S, V, N) \; . \tag{1.40}$$

Using Euler's theorem we can write

$$E = S\frac{\partial E}{\partial S} + V\frac{\partial E}{\partial V} + N\frac{\partial E}{\partial N} \; . \tag{1.41}$$

These partial derivatives can be replaced by intensive variables using first law. As such, the above relation becomes

$$E = TS - PV + \mu N \; . \tag{1.42}$$

This relation is known as **Euler's Relation**.

Thus, the mass of the black hole calculated using this relation becomes

$$M = \frac{r_+}{2} + \frac{3r_+^3}{2l^2} \tag{1.43}$$

which does not agree with the formula given in eq.(1.34). Kastor et al. [92] pointed out that since all dimensionful variables changes under overall scale change, the cosmological constant being dimensionful should change too. If mass $M \equiv M(S, \Lambda)$, and under overall scale change $\Lambda \propto \lambda^{-2}$, then according to Euler's theorem

$$\begin{aligned} M &= 2S\frac{\partial M}{\partial S} - 2\Lambda\frac{\partial M}{\partial \Lambda} \\ &= 2TS - 2\Lambda\frac{\partial M}{\partial \Lambda} \ . \end{aligned} \tag{1.44}$$

The Smarr relation so obtained matches the mass formula given in eq.(1.34). Authors of [92] proposed that in order to get the partial derivative term $\dfrac{\partial M}{\partial \Lambda}$, an extension of the first law is necessary. The proposed form of the extended first law is

$$dM = TdS + \frac{\partial M}{\partial \Lambda} d\Lambda \ . \tag{1.45}$$

Reference [92] further discusses different approaches to derive a variable conjugate to Λ. Λ has also been renamed as pressure in the literature, possibly due to its interpretation as vacuum energy. If we define $P = -\dfrac{\Lambda}{8\pi}$ and its conjugate variable $V = \dfrac{\partial M}{\partial P}$, then it takes a simplified form similar to the volume of a sphere. From eq.(1.34) volume V takes the form

$$V = \frac{4}{3}\pi r_+^3 \ . \tag{1.46}$$

Also, the Smarr relation in these new variables takes the form

$$M = 2TS - 2PV \ . \tag{1.47}$$

It is important to emphasize the fact at this point that the expression of V has no interpretation as the volume of a sphere or the volume enclosed by horizon, as it does not represent any section of the manifold. The presence of the VdP term in the first law suggests that the mass of a black solution should be considered as the enthalpy of the system, rather than its internal energy, in the extended phase space. This new prespective leads to an intricate phase structure, discussed extensively in the inexhaustive list [95–99]. The black holes in this new regime have been shown to exhibit van der Waals fluid-like behaviour [93, 94] similar to the one discussed in case of charged AdS black holes in the previous sections (when analysed at thermodynamic level without any reference to partition function). These show a small-to-large black hole phase transition [93, 94]. However, considering Λ as a variable introduces certain ambiguities in defining the partition function, especially in defining the thermal AdS or extremal black hole background. Disregarding these ambiguities and viewing the black hole purely from a thermodynamic perspective, they conform to the van der Waals equation of state and even demonstrate the same critical exponents [93]. For the RN black hole in the extended phase space, the first law takes the form [93]

$$dM = TdS + \Phi dQ + VdP \ . \tag{1.48}$$

The equation of state as derived in [93] is of the form

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} \ . \tag{1.49}$$

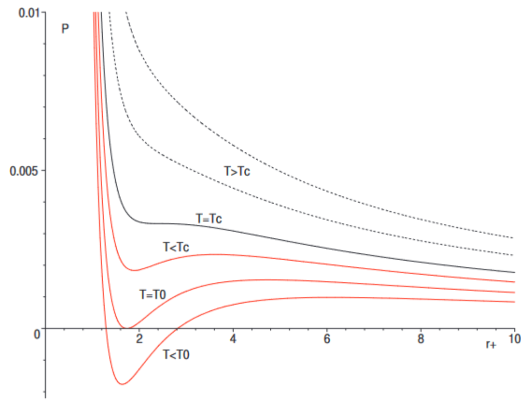


Figure 1.7: P − V diagram of charged AdS black holes for $Q = 1$ **J. High Energ. Phys. 2012, 33 (2012)*

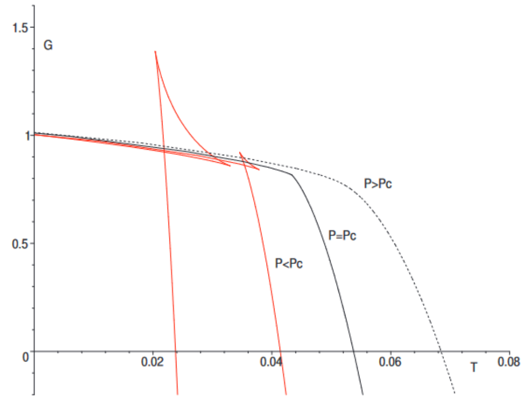


Figure 1.8: Gibbs free energy of charged AdS black hole for $Q = 1$ **J. High Energ. Phys. 2012, 33 (2012)*

The $P - V$ characteristics and Gibbs free energy of the black hole are depicted in Fig.(s)(1.7,1.8) sourced from [93]. The behaviour closely resembles that of a van der Waals fluid and exhibit a small-to-large black hole phase transition below the critical point. Chapter 3 is focused on extended black hole thermodynamics from a novel perspective, incorporating insights from holography in Gauss-Bonnet gravity. We shall also thoroughly investigate the qualitative and quantitative impacts of the Born-Infeld parameter on the phase transition properties.

The thesis is organised as follows: After the basic introduction to the problem of thermal stability and black hole phase transition in this chapter, we move to the first part of the thesis in Chapter 2, which is dedicated to the phase transition in Gauss-Bonnet and Born-Infeld black holes in a constant AdS background. We explore van der Waals fluid-like behaviour and the breaking of critical central charge universality in extended parameter space in chapter 3 (second part of the thesis). We summarize the results and provide a perspective on future projects in chapter 4.

CHAPTER 2

Black Holes Phase Transition in Constant AdS Background

2.1 Introduction

The cosmological constant within Einstein's equation plays a pivotal role, not only in the realm of cosmology but also significantly influences the thermodynamics of black holes. The special case of a negative cosmological constant holds its own importance in the theoretical understanding of black holes from a thermodynamic perspective. Page and Hawking [53] conducted pioneering research on the thermodynamic properties of Schwarzschild black holes in AdS spacetime, unearthing their thermal stability and interesting phase transition properties. The well-behaved characteristics of black holes in AdS spacetime, in general, can be attributed to the gravitational potential in relation to any given point within the space. The potential increases with distance from any point and effectively acts as a confining box for massive particles, thereby ensuring the thermal stability of the black hole.

The examination of black hole phase transition properties in AdS spacetime also carries significant implications within the framework of the gauge/gravity duality [100–103]. Notably, the Hawking-Page phase transition has been demonstrated to explain the confinement-deconfinement phase transition within strongly coupled field theory (QCD), an intricate problem that is challenging to resolve otherwise [102]. Furthermore, the contemporary applications of this duality in the realms of condensed matter physics [104, 105] and cosmology [106–108] has facilitated to deal with systems where perturbative approaches fail. All these phenomena heavily rely on the phase transition properties of black holes in AdS space.

Motivated by Davies's work [43] on Ker-Newman black holes in asymptotically flat spacetime, we consider the study of phase transition properties of Born-Infeld and Gauss-Bonnet black holes in a broader context of general D dimensional AdS spacetime. While the thermodynamics of these black holes have already been considered from different perspectives in literature, our work goes beyond mere heat capacity calculations, aiming to characterise the order of phase transition through Ehrenfest scheme [109] and Ruppeiner state space geometry analysis [110]. Both of these techniques have been widely utilized in the study of black holes [111–123]

The Ehrenfest scheme of phase transition is based on the free energy analysis, where the nth order of phase transition is determined by a discontinuity in its nth order derivative. For a second order phase transition, the continuity of free energy and its first-order derivative on both sides of the phase transition point gives two equations, which are called Ehrenfest's equations. The validity of these two equations by any system confirms the second order nature of phase transition.

On the other hand, Ruppeiner's state space geometry is a geometric explanation of thermodynamics which also goes beyond thermodynamics. This framework introduces a notion of distance by incorporating the theory of fluctuations with standard axioms of thermodynamics. The resulting metric on the thermodynamic manifold is then used to find the curvature of the manifold. The Riemannian curvature of the two dimensional thermodynamic manifold is also called the Ruppeiner curvature and the singularities in the curvature indicate a second-order phase transition.

This chapter is structured as follows: In the initial major section, we begin by examining Born-Infeld black holes in D dimensions, starting with an exploration of their thermodynamic properties and subsequently conducting heat capacity calculations. The succeeding subsections are dedicated to the Ehrenfest analysis and Ruppeiner geometric analysis. The second major section is devoted to Gauss-Bonnet black holes coupled with Born-Infeld electromagnetism. We review their thermodynamic characteristics, followed by an in-depth investigation of the phase structure, through the aforementioned methods.

This chapter is based on the following two publications

- **N. Kumar**, S. Bhattacharyya and S. Gangopadhyay, *Phase transitions in Born-Infeld AdS* *black holes in D-dimensions*, Gen Relativ Gravit (2020) 52:20

- **N. Kumar** and S. Gangopadhyay, *Phase transitions in D-dimensional Gauss–Bonnet–Born–Infeld AdS black holes*, Gen Relativ Gravit 53, 35 (2021)

We start with discussing briefly about the thermodynamic properties of Born-Infeld AdS black holes in the subsequent section.

2.2 D-dimensional Born-Infeld AdS Black Hole and Thermodynamic Properties

We shall give a very brief review of the Born-Infeld AdS black hole solution and its thermodynamic properties. The action in D spacetime dimensions is given by

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g}[R - 2\Lambda + L(F)] \tag{2.1}$$

where the Born-Infeld Lagrangian density is given by $L(F) = 4b^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2b^2}}\right)$. A static black hole solution in general D dimensions with negative cosmological constant value of the form $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$ is given by [81]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{D-2} \; , \tag{2.2}$$

where the lapse function has the form

$$\begin{aligned} f(r) = 1 - \frac{m}{r^{D-3}} + r^2 + \frac{4b^2r^2}{(D-1)(D-2)}\left(1 - \sqrt{1 + \frac{(D-2)(D-3)q^2}{2b^2r^{2D-4}}}\right) \\ + \frac{2(D-2)q^2}{(D-1)r^{2D-6}} {}_2F_1\left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{(D-2)(D-3)q^2}{2b^2r^{2D-4}}\right] \; . \end{aligned} \tag{2.3}$$

We set the value of the AdS radius l to unity, without any loss of generality, leading to the consideration of a black hole within a constant AdS background. As the parameter b approaches infinity, the Reissner-Nordstrom AdS black hole geometry is retrieved. The term ${}_2F_1$ in the above expression represents a hypergeometric function of the form ${}_2F_1[a, b, c, z]$. The parameters m and q in the expression are linked to the ADM mass (M) and the total charge (Q) of the black hole as

$$\begin{aligned} M &= \frac{(D-2)\omega}{16\pi} m \\ Q &= \sqrt{2(D-2)(D-3)} \frac{\omega}{8\pi} q \, , \qquad \omega = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma \frac{(D-1)}{2}} \, . \end{aligned} \tag{2.4}$$

The ADM mass of the black hole can be written in terms of horizon radius (r_+) using condition $f(r_+) = 0$. The form the ADM mass takes is

$$\begin{aligned} M = \frac{(D-2)\omega}{16\pi} r_+^{D-3} + \frac{(D-2)\omega}{16\pi} r_+^{D-1} + \frac{b^2\omega}{4\pi(D-1)} r_+^{D-1} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2\omega^2 r_+^{2D-4}}}\right) \\ + \frac{4\pi(D-2)Q^2}{(D-1)(D-3)\omega r_+^{D-3}} {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi Q^2}{b^2\omega^2 r_+^{2D-4}}\right] . \end{aligned} \tag{2.5}$$

Also, the Hawking temperature of the black hole can be calculated from the lapse function in the following manner

$$\begin{aligned} T &= \frac{1}{4\pi} \left(\frac{df(r)}{dr}\right)_{r_+} \\ &= \frac{1}{4\pi} \left[\frac{D-3}{r_+} + (D-1) r_+ + \frac{4b^2 r_+}{D-2} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2\omega^2 r_+^{2D-4}}}\right)\right] . \end{aligned} \tag{2.6}$$

The black hole is parameterised by two variables, mass (M) and charge (Q). Hence, the first law of black hole thermodynamics will be of the form

$$dM = TdS + \Phi dQ \, . \tag{2.7}$$

In analogy to the first law of thermodynamics,

$$dU = TdS - PdV, \tag{2.8}$$

we can identify the electrostatic potential Φ with the negative of the pressure and charge by volume. The mass of the black hole is equivalent to the internal energy U of the system. The analogy extablished at this moment will be used later to identify other thermodynamic quantities. It is important to note that the analogy is purely mathematical and should not be stretched for any interpretational identification. Using relation (2.7), the black hole entropy S can be calculated as

$$S = \int_0^{r_+} \frac{1}{T} \left(\frac{\partial M}{\partial r} \right)_Q dr = \frac{\omega}{4} r_+^{D-2} \,, \tag{2.9}$$

which is the standard Bekenstein-Hawking formula. The gauge potential Φ for the black hole can also be calculated from the first law. It takes the form

$$\Phi = \frac{q}{c} \frac{1}{r_+^{D-3}} \, {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{(D-2)(D-3)q^2}{2b^2 r_+^{2D-4}} \right] \,, \tag{2.10}$$

where $c = \sqrt{\dfrac{2(D-3)}{D-2}}$. Now, we can express the temperature and gauge potential in terms of entropy by substituting r_+ with S in eq.(s)(2.6,2.10). These take the form

$$\begin{aligned} T = \frac{1}{4\pi} \Bigg[& (D-3) \left(\frac{\omega}{4S} \right)^{\frac{1}{D-2}} + (D-1) \left(\frac{4S}{\omega} \right)^{\frac{1}{D-2}} \\ & + \frac{4b^2}{(D-2)} \left(\frac{4S}{\omega} \right)^{\frac{1}{D-2}} \left(1 - \sqrt{1 + \frac{\pi^2 Q^2}{b^2 S^2}} \right) \Bigg] \end{aligned} \tag{2.11}$$

and

$$\Phi = \frac{4\pi Q}{\omega(D-3)} \left(\frac{\omega}{4S} \right)^{\frac{D-3}{D-2}} \, {}_2F_1 \left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{\pi^2 Q^2}{b^2 S^2} \right] \,. \tag{2.12}$$

We illustrated the relationship between the Hawking temperature and entropy in Fig. (s) (2.1, 2.2, 2.3) for a specific parameter set, for spacetime dimensions 4, 5, and 6. Notably, the plots exhibit a continuous trend. Initially, the temperature experiences an incline for small entropy values, corresponding to smaller-sized black holes. Subsequently, it declines, reaching a minimum before increasing again as the black hole size expands. The inverse relationship between temperature and entropy observed here deviates from the conventional behavior of typical thermodynamic entities. However, a direct analysis suggests the absence of any first-order phase transition as entropy is a first-order derivative of free energy.

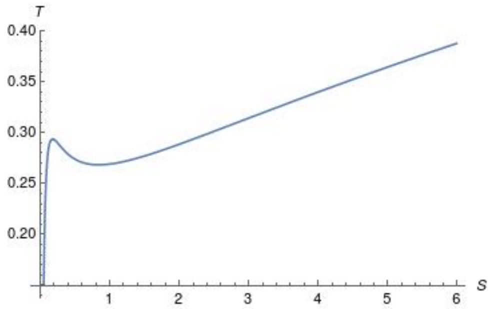


Figure 2.1: Temperature vs Entropy: D=4, Q=0.13, b=10

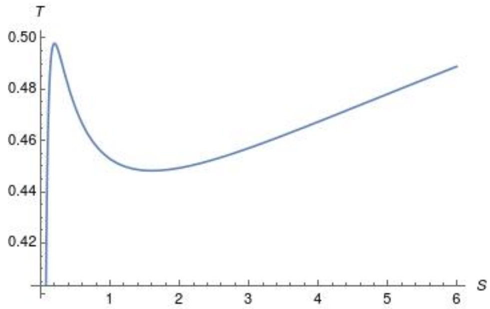


Figure 2.2: Temperature vs Entropy: D=5, Q=0.13, b=10

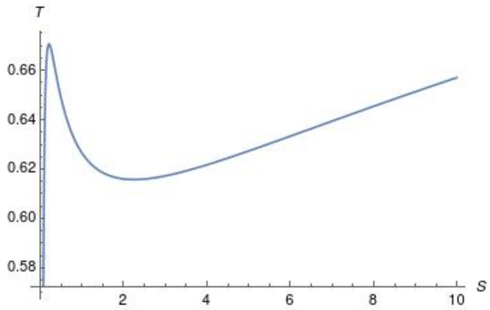


Figure 2.3: Temperature vs Entropy: D=6, Q=0.13, b=10

Next, we calculate the specific heat capacity (C_Φ) at constant electric potential, which is analogous to the heat capacity at constant pressure. The heat capacity is given by

$$C_\Phi = T\left(\frac{\partial S}{\partial T}\right)_\Phi \, . \tag{2.13}$$

To calculate it, we can safely consider the black hole temperature as a function of entropy and electric charge, leading to the following relation:

$$\left(\frac{\partial T}{\partial S}\right)_\Phi = \left(\frac{\partial T}{\partial S}\right)_Q - \left(\frac{\partial T}{\partial Q}\right)_S \left(\frac{\partial \Phi}{\partial S}\right)_Q \left(\frac{\partial Q}{\partial \Phi}\right)_S \tag{2.14}$$

where we used the following thermodynamic identity

$$\left(\frac{\partial Q}{\partial S}\right)_\Phi \left(\frac{\partial S}{\partial \Phi}\right)_Q \left(\frac{\partial \Phi}{\partial Q}\right)_S = -1 \, . \tag{2.15}$$

The heat capacity C_Φ can be calculated now using eq.(s)(2.11), (2.12) and (2.14). The expression takes the form

$$C_\Phi = \frac{\mathcal{N}(Q,b,S,\omega)}{\mathcal{M}(Q,b,S,\omega)} \ , \tag{2.16}$$

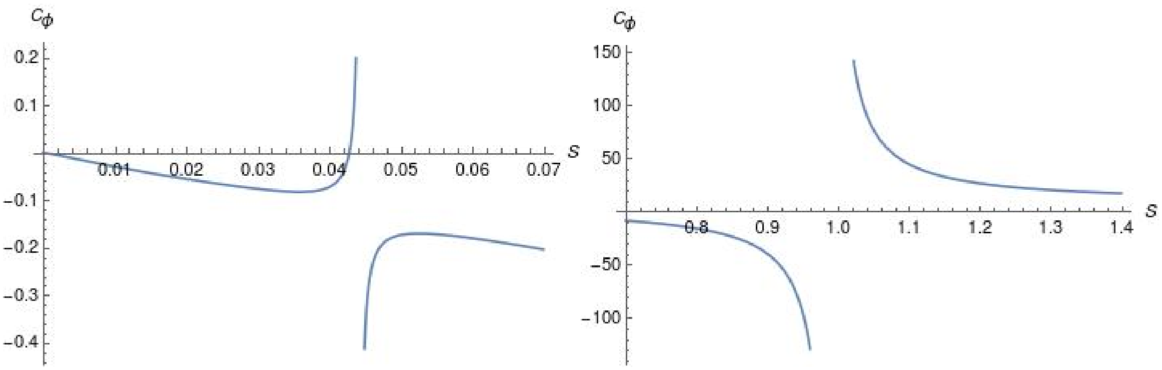
where

$$\begin{aligned}\mathcal{N}(Q,b,S,\omega) = ((D-2)S^3)&\left(D-3+\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2Q^2}{b^2S^2}\right]\right)\\ &\times\left[\left(\frac{S}{\omega}\right)^{\frac{1}{D-2}}2^{\frac{4}{D-2}}\left(D^2-3D+2+\right.\right.\\ &\left.\left.4b^2\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\right)+\left(\frac{\omega}{S}\right)^{\frac{1}{D-2}}(D^2-5D+6)\right]\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&\mathcal{M}(Q,b,S,\omega)\\ &\quad= S^2\left(\frac{S}{\omega}\right)^{\frac{1}{D-2}}2^{\frac{4}{D-2}}\left(D^3-6D^2+11D-6+4b^2(D-3)\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\right)\\ &\quad- S^2\times\left(\frac{\omega}{S}\right)^{\frac{1}{D-2}}(D^3-8D^2+21D-18)\\ &\quad+\left(\frac{S}{\omega}\right)^{\frac{1}{D-2}}2^{\frac{4}{D-2}}\times 4\pi^2Q^2(D-3){}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2Q^2}{b^2S^2}\right]\\ &\quad+S^2\left(\frac{S}{\omega}\right)^{\frac{1}{D-2}}2^{\frac{4}{D-2}}\left[(D^2-3D+2+4b^2)\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}-4b^2\right]\\ &\qquad\quad\times{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2Q^2}{b^2S^2}\right]\\ &\quad-S^2\left(\frac{\omega}{S}\right)^{\frac{1}{D-2}}\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}(D^2-5D+6){}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2Q^2}{b^2S^2}\right]\ .\end{aligned} \tag{2.18}$$

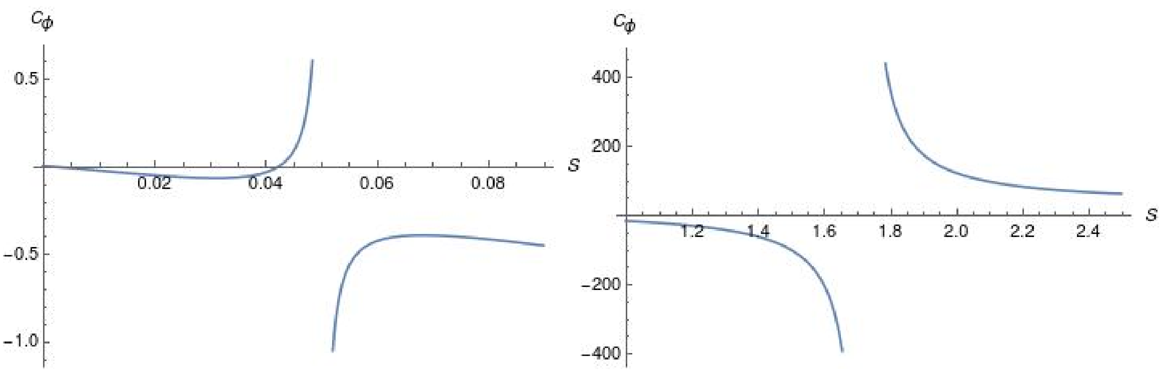
Figure (s) (2.4 ,..., 2.6) depicts the relationship between the heat capacity C_Φ and entropy across various dimensions, using a fixed set of parameters. Across all dimensions, the heat capacity displays two points of infinite discontinuity, denoted as S_1 and S_2, marking critical points that signify phase transitions. The observed behavior closely aligns with the predictions from the T-S plot. The system exhibits three distinct phases: Phase I ($0 < S < S_1$), Phase II ($S_1 < S < S_2$), and Phase III ($S > S_2$). Phase I is thermodynamically stable (apparent negative values of heat capacity belong to unphysical black hole region where temperature is negative), as indicated by the positive heat capacity, while Phase II is unstable, and Phase III is stable once more. These phases undergo transitions at the critical junctions and are respectively termed the small stable black hole phase (SSB), intermediate unstable black hole phase (IUB), and large stable black hole phase (LSB). Notably, the qualitative behavior bears resemblance to that of the Reissner-Nordstrom black holes [111]. Our findings also align with the results previously established in [111] for $D = 4$.



(a) D=4, Q=0.13, b=10, Critical point (S_1)

(b) D=4, Q=0.13, b=10, Critical point (S_2)

Figure 2.4: Heat Capacity vs Entropy

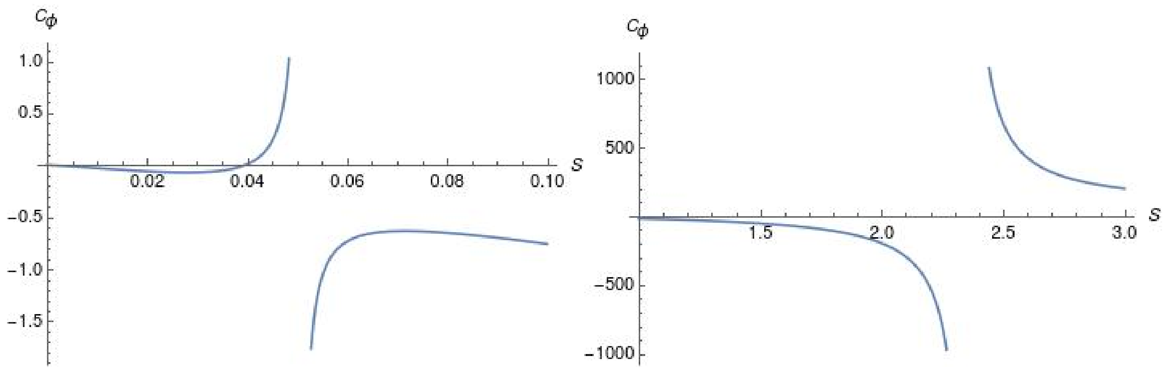


(a) D=5, Q=0.13, b=10, Critical point (S_1)

(b) D=5, Q=0.13, b=10, Critical point (S_2)

Figure 2.5: Heat Capacity vs Entropy

Exact values of phase transition points are tabulated in Table 2.1.



(a) D=6, Q=0.13, b=10, Critical point (S_1) (b) D=6, Q=0.13, b=10, Critical point (S_2)

Figure 2.6: Heat Capacity vs Entropy

	D=4	D=5	D=6
S_1	0.0443255	0.0502572	0.0506412
S_2	0.991135	1.72042	2.35452

Table 2.1: Values of entropy where C_Φ diverge.

While the precise order of the phase transition between stable and unstable phases remains unclear at this point, we can confidently assert the absence of a first-order phase transition, as evidenced by the continuous T-S plot. Furthermore, the observed discontinuity in the heat capacity plots indicates a phase transition of an order greater than one. To determine the specific order of the phase transition, we will employ the Ehrenfest scheme, a technique that aligns with free energy analysis. Additionally, we will utilize the Ruppeiner state space analysis, a geometric approach, to further validate and confirm the order.

2.2.1 Order of Phase Transition using Ehrenfest Scheme

Free energy analysis facilitates figuring out the order of phase transition. The change is considered a first order phase transition if the first order derivative of the free energy is discontinous. Similarly, the phase transition is of nth order if the nth derivative of the free energy is discontinous. Using this concept, the continuity of the free energy and its first order derivative on both sides of the phase transition point yields two equations known as the Ehrenfest equations for a second order phase transition.

These two equations in standard thermodynamics are given by

$$\left(\frac{\partial P}{\partial T}\right)_S = \frac{1}{VT}\frac{C_{P_2}-C_{P_1}}{\beta_2-\beta_1} = \frac{\Delta C_P}{VT\Delta\beta}, \tag{2.19}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} = \frac{\Delta\beta}{\Delta\kappa}\ . \tag{2.20}$$

Here, subscripts 1 and 2 denote two sides of the phase transition point. For the black hole under consideration, the pressure (P) is represented by the negative of the electrostatic potential difference (Φ) between the horizon and asymptotic infinity, and the volume (V) is represented by the charge (Q) on the black hole. Thus, these two equations for black hole system become

$$-\left(\frac{\partial \Phi}{\partial T}\right)_S = \frac{1}{QT}\frac{C_{\Phi_2}-C_{\Phi_1}}{\beta_2-\beta_1} = \frac{\Delta C_\Phi}{QT\Delta\beta} \tag{2.21}$$

$$-\left(\frac{\partial \Phi}{\partial T}\right)_Q = \frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} = \frac{\Delta\beta}{\Delta\kappa}\ . \tag{2.22}$$

Here, β and κ are the volume expansion coefficient and isothermal compressibility of the system, respectively. For the black hole system, these are defined as

$$\beta = \frac{1}{Q}\left(\frac{\partial Q}{\partial T}\right)_\Phi\ , \quad \kappa = \frac{1}{Q}\left(\frac{\partial Q}{\partial \Phi}\right)_T\ . \tag{2.23}$$

The validity of these two equations will confirm the second order nature of phase transition observed in the discontinuities in heat capacity at points S_i (i=1,2). Rest of the parameters will also be denoted by subscript i at the points of phase transition. For example, temperature and charge at the phase transition points will be T_i and Q_i. Now, we proceed to check the validity of these equations.

By chain rule, the left hand side of the first Ehrenfest equation (2.21), at the phase transition points can be expanded as

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} = -\left[\left(\frac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i}\left[\left(\frac{\partial Q}{\partial T}\right)_S\right]_{S=S_i}$$

$$= \; -\frac{\left[\left(\dfrac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i}}{\left[\left(\dfrac{\partial T}{\partial Q}\right)_S\right]_{S=S_i}} \; . \tag{2.24}$$

Also, using eq.(2.23) and heat capacity definition $(C_\Phi = T(\frac{\partial S}{\partial T})_\Phi)$, the right hand side of eq.(2.21) can be written as

$$Q_i\beta = \left[\left(\frac{\partial Q}{\partial T}\right)_\Phi\right]_{S=S_i} = \left[\left(\frac{\partial Q}{\partial S}\right)_\Phi\right]_{S=S_i}\left(\frac{C_\Phi}{T_i}\right) \tag{2.25}$$

$$\frac{\Delta C_\Phi}{T_i Q_i \Delta\beta} = \left[\left(\frac{\partial S}{\partial Q}\right)_\Phi\right]_{S=S_i} \; . \tag{2.26}$$

Use of identity (2.15), in the above equation leads to

$$\frac{\Delta C_\Phi}{T_i Q_i \Delta\beta} = -\frac{\left[\left(\dfrac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i}}{\left[\left(\dfrac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i}} \; . \tag{2.27}$$

Thus, from eq.(s) (2.21, 2.24, 2.27), the simplified form of the first Ehrenfest equation becomes

$$\left[\left(\frac{\partial T}{\partial Q}\right)_S\right]_{S=S_i} = \left[\left(\frac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i} \; . \tag{2.28}$$

Equality of the above two differentials will give the validity of first Ehrenfest equation. The left hand side of the above equation can be obtained by differentiating eq.(2.11). The expression takes the form

$$\left[\left(\frac{\partial T}{\partial Q}\right)_S\right]_{S=S_i} = -\frac{\pi Q_i}{(D-2)S_i^2}\left(\frac{4S_i}{\omega}\right)^{\dfrac{1}{D-2}}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{-\dfrac{1}{2}} \tag{2.29}$$

and using eq.(2.12), the right hand side of eq.(2.28) becomes

$$\left[\left(\frac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i} = -\frac{\pi Q_i}{(D-2)S_i^2}\left(\frac{4S_i}{\omega}\right)^{\dfrac{1}{D-2}}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{-\dfrac{1}{2}} \; . \tag{2.30}$$

The above two equations show that the black hole follows the first Ehrenfest's equation. The expression of the differential in numerator of eq.(s)(2.24,2.27), $\left[\left(\frac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i}$ becomes

$$\left[\left(\frac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i} = \frac{4\pi}{\omega(D-2)(D-3)}\left(\frac{\omega}{4S_i}\right)^{\frac{D-3}{D-2}}\left[(D-3)\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{-\frac{1}{2}} + {}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right]\right] . \quad (2.31)$$

Therefore, the forms of eq.(2.24) and eq.(2.27) become

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{S_i}{Q_i}\left[1+\frac{1}{D-3}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{\frac{1}{2}} {}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right]\right] \quad (2.32)$$

and

$$\frac{\Delta C_\Phi}{T_i Q_i \Delta\beta} = \frac{S_i}{Q_i}\left[1+\frac{1}{D-3}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{\frac{1}{2}} {}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right]\right] , \quad (2.33)$$

respectively. It is also reassuring to find that in $D=4$, eq.(s)(2.32, 2.33) reduce to the results in [111] which is given by

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} = \frac{\Delta C_\Phi}{T_i Q_i \Delta\beta} = \frac{S_i}{Q_i}\left[1+\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{\frac{1}{2}} {}_2F_1\left[\frac{1}{4},\frac{1}{2},\frac{5}{4},-\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right]\right] . \quad (2.34)$$

We now move to checking the validity of second Ehrenfest equation (2.22). If we consider temperature as a function of entropy and electric potential, that is,

$$T \equiv T(S,\Phi),$$

then

$$\left(\frac{\partial T}{\partial \Phi}\right)_Q = \left(\frac{\partial T}{\partial S}\right)_\Phi \left(\frac{\partial S}{\partial \Phi}\right)_Q + \left(\frac{\partial T}{\partial \Phi}\right)_S . \quad (2.35)$$

At the critical point, the heat capacity diverges, hence $\left[\left(\frac{\partial T}{\partial S}\right)_{\Phi}\right]_{S=S_i} = 0$. From eq.(2.30), it is clear that $\left(\frac{\partial S}{\partial \Phi}\right)_Q$ is finite at the critical point, therefore

$$\left[\left(\frac{\partial T}{\partial \Phi}\right)_Q\right]_{S=S_i} = \left[\left(\frac{\partial T}{\partial \Phi}\right)_S\right]_{S=S_i} \tag{2.36}$$

which implies

$$\begin{aligned}\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} &= \left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} \\ &= -\frac{S_i}{Q_i}\left[1+\frac{1}{D-3}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{\frac{1}{2}} {}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right]\right]\end{aligned} \tag{2.37}$$

Also, eq.(2.23) at the phase transition points can be written as

$$\kappa Q_i = \left[\left(\frac{\partial Q}{\partial \Phi}\right)_T\right]_{S=S_i} . \tag{2.38}$$

If we use the identity $\left(\frac{\partial Q}{\partial \phi}\right)_T \left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_{\Phi} = -1$, and the definition of β in eq.(2.23) for the black hole system, we find

$$\kappa Q_i = \left[\left(\frac{\partial T}{\partial \Phi}\right)_Q\right]_{S=S_i} Q_i \beta . \tag{2.39}$$

Thus, the right hand side of the second Ehrenfest equation (2.22) becomes

$$\frac{\Delta \beta}{\Delta \kappa} = -\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} . \tag{2.40}$$

The above equation can also be written as

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} = -\left[\left(\frac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i}\left[\left(\frac{\partial S}{\partial T}\right)_Q\right]_{S=S_i}$$

$$= -\frac{\left[\left(\dfrac{\partial\Phi}{\partial S}\right)_Q\right]_{S=S_i}}{\left[\left(\dfrac{\partial T}{\partial S}\right)_Q\right]_{S=S_i}} . \tag{2.41}$$

From eq.(2.11), we have

$$\left[\left(\frac{\partial T}{\partial S}\right)_Q\right]_{S=S_i} = \frac{1}{4\pi}\left[-\frac{(D-3)}{(D-2)S_i}\left(\frac{\omega}{4S_i}\right)^{\dfrac{1}{D-2}} + \frac{(D-1)}{(D-2)S_i}\left(\frac{4S_i}{\omega}\right)^{\dfrac{1}{D-2}}\frac{4b^2}{(D-2)^2 S}\right.$$
$$\left.\times\left(\frac{4S_i}{\omega}\right)^{\dfrac{1}{D-2}}\left(1-\sqrt{1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}}\right) + \frac{4\pi^2 Q_i^2}{(D-2)S^3}\left(\frac{4S_i}{\omega}\right)^{\dfrac{1}{D-2}}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{-\dfrac{1}{2}}\right] . \tag{2.42}$$

Using eq.(s)(2.30, 2.42), eq.(2.41) becomes

$$-\left[\left(\frac{\partial\Phi}{\partial T}\right)_Q\right]_{S=S_i} = \frac{S_i}{Q_i}\left[1+\frac{1}{D-3}\left(1+\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right)^{\dfrac{1}{2}} {}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2 Q_i^2}{b^2 S_i^2}\right]\right] . \tag{2.43}$$

Thus, the equality of eq.(s)(2.37, 2.43) prove that the second Ehrenfest equation (2.22) is also satisfied at the critical points.

The reliability of these two equations unambiguously demonstrates the precise second-order nature of the phase transition. In the upcoming section, we will employ a technique of thermodynamic geometry which is widely trusted in both standard and black hole thermodynamics.

2.2.2 Ruppeiner state space geometry and Phase Transition

Geometric methods have long been instrumental in the analysis of thermodynamic systems, resulting in comprehensive exploration within this field [110, 115–123]. The introduction of geometric concepts relies on the definition of a line element, combining fluctuation theory with the fundamental postulates of equilibrium thermodynamics. The connection between two states is established through the fluctuation probability, with less probable fluctuating states characterized by longer distances and more probable states by shorter distances. Research has demonstrated that the computation of scalar curvature within this geometric framework provides valuable insights into the distinct phases and the transitional behavior that occurs between them. The literature discusses two geometries based on the choice of potential used to define the metric. In the context of Ruppeiner geometry [110], entropy serves as the thermodynamic potential, while the Weinhold metric [124] relies on internal energy. In the previous section, we conducted an Ehrenfest analysis at the critical points, identifying these points as the locations of second-order phase transitions. In this section, we turn our attention to the examination of points with infinite discontinuities using the Ruppeiner geometry. To delve deeper into this analysis, we proceed with the calculation of the Ruppeiner curvature (R) for the thermodynamic manifold. The Ruppeiner metric coefficients for the thermodynamic parameter space are given by [110]

$$g_{ij}^{R} = -\frac{\partial^2 S(x^i)}{\partial x^i \partial x^j} \tag{2.44}$$

where $x^i = x^i(M, Q)$; i=1,2. It is convenient to use mass of the black hole as a thermodynamic potential. For which, we shall use the Weinhold metric coefficients, which are defined as

$$g_{ij}^{W} = \frac{\partial^2 M(x^i)}{\partial x^i \partial x^j} \tag{2.45}$$

where $x^i = x^i(S, Q)$; i=1,2. The two geometries are connected to each other by the following relation

$$dS_R^2 = \frac{dS_W^2}{T} \ . \tag{2.46}$$

The mass of the black hole can be expressed in terms of the entropy by replacing horizon radius with entropy. The form of the expression becomes

$$\begin{aligned} M(S,Q) = {} & \frac{(D-2)\omega}{16\pi}\left(\frac{4S}{\omega}\right)^{\frac{D-3}{D-2}} + \left(\frac{4S}{\omega}\right)^{\frac{D-1}{D-2}}\left[\frac{(D-2)\omega}{16\pi} + \frac{b^2\omega}{4\pi(D-1)}\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\right] \\ & + \frac{4\pi(D-2)}{\omega(D-1)(D-3)}Q^2\left(\frac{\omega}{4S}\right)^{\frac{D-3}{D-2}} {}_2F_1\left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{\pi^2Q^2}{b^2S^2}\right] \ . \end{aligned} \tag{2.47}$$

We will employ the aforementioned expression to compute the Ruppeiner metric coefficients. Upon taking the second derivative with respect to the extensive variables, we obtain the metric coefficients as follows:

$$
\begin{aligned}
g^R_{SS} = \frac{1}{T}\Bigg[&-\frac{(D-3)\omega}{16\pi(D-2)}\left(\frac{4}{\omega}\right)^{\frac{D-3}{D-2}} S^{-\frac{D-1}{D-2}} + \frac{(D-1)\omega}{16\pi(D-2)}\left(\frac{4}{\omega}\right)^{\frac{D-1}{D-2}} S^{-\frac{D-3}{D-2}} \\
&+\frac{b^2\omega}{4\pi(D-2)^2}\left(\frac{4}{\omega}\right)^{\frac{D-1}{D-2}} S^{-\frac{D-3}{D-2}}\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right) \\
&+\frac{\omega\pi Q^2}{4(D-2)S^3}\left(\frac{4}{\omega}\right)^{\frac{D-1}{D-2}} S^{\frac{1}{D-2}}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}}\Bigg]
\end{aligned}
\tag{2.48}
$$

$$
g^R_{SQ} = \frac{1}{T}\left[-\frac{\omega\pi Q}{4(D-2)S^2}\left(\frac{4}{\omega}\right)^{\frac{D-1}{D-2}} S^{\frac{1}{D-2}}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}}\right] \tag{2.49}
$$

$$
\begin{aligned}
g^R_{QQ} = \frac{1}{T}\Bigg(&\frac{4\pi}{\omega(D-3)}\left(\frac{\omega}{4S}\right)^{\frac{D-3}{D-2}}\Bigg[\frac{D-3}{D-2}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}} \\
&+\frac{1}{D-2}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{\pi^2Q^2}{b^2S^2}\right]\Bigg]\Bigg)
\end{aligned}
\tag{2.50}
$$

The Ruppeiner curvature is the Riemannian curvature of the thermodynamic manifold which is calculated from the above metric coefficients using the following relation

$$
\begin{aligned}
R =& -\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial S}\left(\frac{g_{SQ}}{\sqrt{g}g_{SS}}\frac{\partial g_{SS}}{\partial Q}-\frac{1}{\sqrt{g}}\frac{g_{QQ}}{\partial S}\right)+\frac{\partial}{\partial Q}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{SQ}}{\partial Q}-\frac{1}{\sqrt{g}}\frac{\partial g_{SS}}{\partial Q}-\frac{g_{SQ}}{\sqrt{g}g_{SS}}\frac{\partial g_{SS}}{\partial S}\right)\right] \\
=& \frac{A(Q,S)}{B(Q,S)}
\end{aligned}
\tag{2.51}
$$

where

$$A(Q,S) = -\left[\frac{\partial}{\partial S}\left(\frac{g_{SQ}}{\sqrt{g}g_{SS}}\frac{\partial g_{SS}}{\partial Q} - \frac{1}{\sqrt{g}}\frac{g_{QQ}}{\partial S}\right) + \frac{\partial}{\partial Q}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{SQ}}{\partial Q} - \frac{1}{\sqrt{g}}\frac{\partial g_{SS}}{\partial Q} - \frac{g_{SQ}}{\sqrt{g}g_{SS}}\frac{\partial g_{SS}}{\partial S}\right)\right]$$

and $B(Q,S) = \sqrt{g}$.

The expression of R is too large, however, the denominator $B(Q,S)$ is of the form

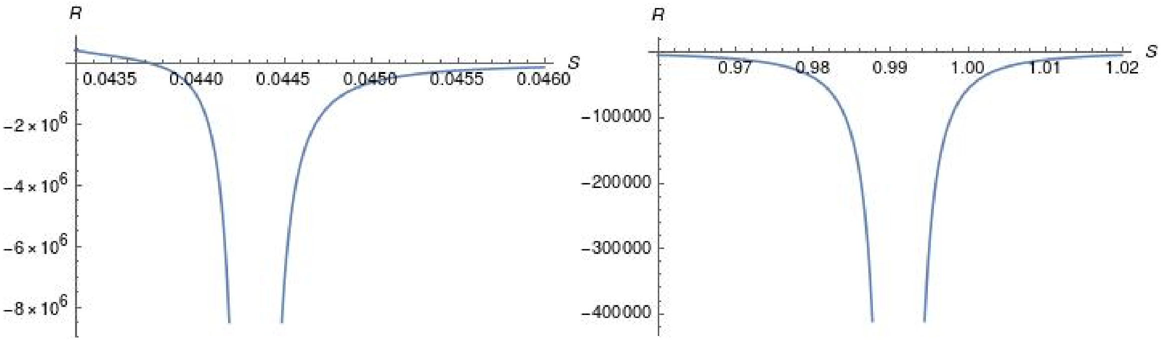
$$\begin{aligned} B(Q,S) = \frac{1}{T}\Bigg[& \frac{\pi^2 Q^2}{(D-2)^2 S^3}\left(\frac{4}{\omega}\right)^{\frac{2}{D-2}} S^{-\frac{D-4}{D-2}}\left[\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-1} + \frac{1}{D-3}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}}\boldsymbol{F}\right] \\ & +\frac{b^2}{(D-2)^3}\left(\frac{4}{\omega}\right)^{\frac{2}{D-2}} S^{-\frac{2(D-3)}{D-2}}\left(1-\sqrt{1+\frac{\pi^2Q^2}{b^2S^2}}\right)\left[\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}} + \frac{1}{D-3}\boldsymbol{F}\right] \\ & +\frac{D-1}{4(D-2)^2}\left(\frac{4}{\omega}\right)^{\frac{2}{D-2}} S^{-\frac{2(D-3)}{D-2}}\left[\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}} + \frac{1}{D-3}\boldsymbol{F}\right] \\ & -\frac{1}{4(D-2)^2S^2}\left[(D-3)\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}} + \boldsymbol{F}\right] \\ & -\frac{\omega^2\pi^2Q^2}{16(D-2)^2S^4}\left(\frac{4}{\omega}\right)^{\frac{2(D-1)}{D-2}} S^{\frac{2}{D-2}}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-1}\Bigg]^{\frac{1}{2}} \end{aligned} \tag{2.52}$$

where $\boldsymbol{F} \equiv {}_2F_1\left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{\pi^2Q^2}{b^2S^2}\right]$.

For spacetime dimensions $D = 4$, the above expression reduces to

$$\begin{aligned} B(Q,S) \equiv \sqrt{g} = \frac{1}{T}\Bigg[& \left(\boldsymbol{F} + \left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}}\right)\left(\frac{3S-\pi}{16\pi S^2} + \frac{b^2}{8\pi S}\left(1-\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{\frac{1}{2}}\right)\right. \\ & \left. +\frac{\pi Q^2}{4S^3}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-\frac{1}{2}}\right) - \frac{\pi Q^2}{4S^3}\left(1+\frac{\pi^2Q^2}{b^2S^2}\right)^{-1}\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{2.53}$$

The roots of $B(Q,S)$ represent the points of phase transition. We illustrated the Ruppeiner curvature for various spacetime dimensions ($D = 4, 5, 6$) with the same parameter values ($Q = 0.13$ and $b = 10$) in Fig. (s) (2.7, 2.8, 2.9). Notably, the Ruppeiner curvature exhibits infinite discontinuity at the points where the heat capacity diverges, providing further confirmation that the phase transition is of second order. The points of infinite discontinuity in the Ruppeiner curvature are presented in Table 2.2.



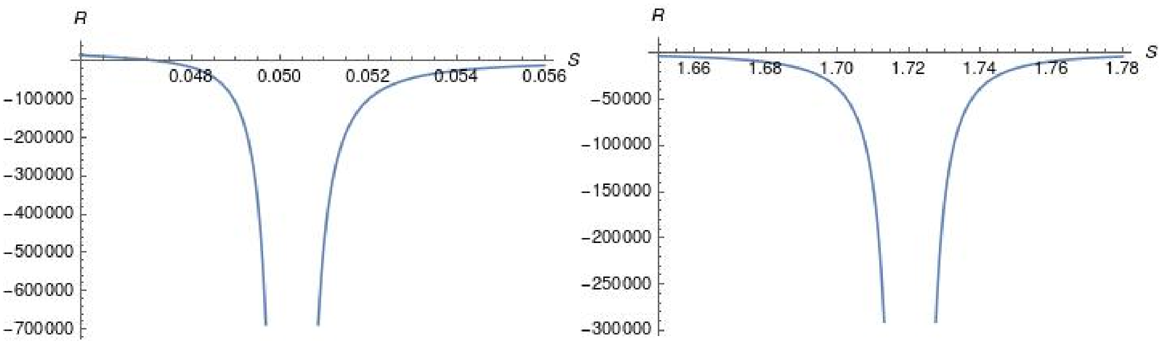
(a) D=4, Q=0.13, b=10, Critical point (S_1) (b) D=4, Q=0.13, b=10, Critical point (S_2)

Figure 2.7: Ruppeiner Curvature vs Entropy

	D=4	D=5	D=6
S_1	0.0443255	0.0502572	0.0506412
S_2	0.991135	1.72042	2.35452

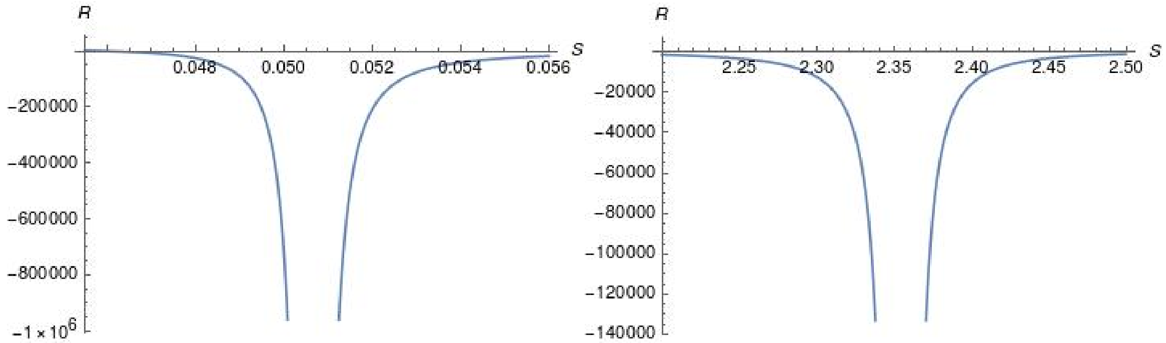
Table 2.2: Values of entropy where R diverge

Subsequently, we will delve into the examination of the phase transition properties of Gauss-Bonnet-Born-Infeld black holes in the following section.



(a) D=5, Q=0.13, b=10, Critical point (S_1) (b) D=5, Q=0.13, b=10, Critical point (S_2)

Figure 2.8: Ruppeiner Curvature vs Entropy



(a) D=6, Q=0.13, b=10, Critical point (S_1)   (b) D=6, Q=0.13, b=10, Critical point (S_2)

Figure 2.9: Ruppeiner Curvature vs Entropy

2.3 Gauss-Bonnet-Born-Infeld AdS Black Holes and Thermodynamic Properties

This section will focus on the thermodynamics of Gauss-Bonnet black holes in the presence of Born-Infeld electromagnetic fields within AdS spacetime. We will provide a concise review of their thermodynamic properties and analyze the black hole solution across general spacetime dimensions, denoted as D. The action of the Gauss-Bonnet gravity with Born-Infeld electromagnetic fields can be represented as follows

$$I = \frac{1}{16\pi}\int d^D x\sqrt{-g}[R - 2\Lambda + \alpha L_{GB} + L(F)]\ . \tag{2.54}$$

Here, the units are $G = c = 1$. The Gauss-Bonnet Lagrangian density is given by $L_{GB} = R^2 - 4R_{\gamma\delta}R^{\gamma\delta} + R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma}$, and α represents the Gauss-Bonnet parameter. The Born-Infeld term retains the same structure as in the previous section, $L(F) = 4b^2\left(1 - \sqrt{1 + \dfrac{F^{\mu\nu}F_{\mu\nu}}{2b^2}}\right)$, where b is the Born-Infeld parameter. The cosmological constant Λ, related to the AdS radius as $\Lambda = -\dfrac{(D-1)(D-2)}{2l^2}$, also contributes to the overall equation. Our primary focus will be on black hole solutions exhibiting non-trivial metric structures, specifically in dimensions greater than 4.

The static, spherically symmetric solution of the following form of the above considered gravity theory reads

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{D-2}^2, \tag{2.55}$$

where the lapse function $f(r)$ is of the form [125]

$$f(r) = 1 + \frac{r^2}{2\alpha'}(1 - \sqrt{g(r)}) \tag{2.56}$$

with $g(r)$ being

$$\begin{aligned} g(r) = 1 - \frac{4\alpha'}{l^2} + \frac{4\alpha' m}{r^{D-1}} - \frac{16\alpha' b^2}{(D-1)(D-2)}\left(1 - \sqrt{1 + \frac{(D-2)(D-3)q^2}{2b^2 r^{2D-4}}}\right) \\ -\frac{8(D-2)\alpha' q^2}{(D-1)r^{2D-4}}{}_2F_1\left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{(D-2)(D-3)q^2}{2b^2 r^{2D-4}}\right]. \end{aligned} \tag{2.57}$$

The parameters m, q and α' are connected to the actual black hole parameters - the ADM mass M, the charge Q and the Gauss-Bonnet coefficent α - in the following manner

$$\begin{aligned} M &= \frac{(D-2)\omega}{16\pi}m, \qquad \alpha = \frac{\alpha'}{(D-3)(D-4)}, \\ Q &= \sqrt{2(D-2)(D-3)}\frac{\omega}{8\pi}q, \qquad \omega = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\frac{(D-1)}{2}}. \end{aligned} \tag{2.58}$$

The ADM mass can be expressed in terms of the horizon radius (r_+) using the condition $f(r_+) = 0$. The expression of the black hole mass is as follows

$$\begin{aligned} M = \frac{(D-2)\omega\alpha'}{16\pi}r_+^{D-5} + \frac{(D-2)\omega}{16\pi}r_+^{D-3} + \frac{(D-2)\omega}{16\pi l^2}r_+^{D-1} \\ + \frac{b^2\omega}{4\pi(D-1)}r_+^{D-1}\left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2\omega^2 r_+^{2D-4}}}\right) \\ + \frac{4(D-2)\pi Q^2}{(D-1)(D-3)\omega r_+^{D-3}}{}_2F_1\left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi^2 Q^2}{b^2\omega^2 r_+^{2D-4}}\right]. \end{aligned} \tag{2.59}$$

The black hole temperature can be calculated using eq.(2.6) as

$$
\begin{aligned}
T &= \frac{1}{4\pi}\left(\frac{df(r)}{dr}\right)_{r_+} \\
&= \frac{1}{4\pi r_+(r_+^2+2\alpha')}\left[\frac{(D-1)r_+^4}{l^2}+(D-3)r_+^2+(D-5)\alpha'\right. \\
&\quad \left. + \frac{4b^2r_+^4}{D-2}\left(1-\sqrt{1+\frac{16\pi^2Q^2}{b^2\omega^2 r_+^{2D-4}}}\right)\right] \; . \qquad (2.60)
\end{aligned}
$$

In both the expression of ADM mass and Hawking temperature, the AdS radius radius, l, has been set to unity without any loss of generalization, as we are interested in black hole thermodynamics in constant AdS background.

Let us now consider the first law of thermodynamics for a charged black hole, which takes the form

$$
dM = TdS + \Phi dQ \qquad (2.61)
$$

As previously demonstrated in the case of the Born-Infeld black hole within Einstein's gravity in the preceding section, we can draw an analogy with the first law in standard thermodynamics, given by

$$
dU = TdS - PdV \qquad (2.62)
$$

Here, we identify the pressure P with the negative of the electrostatic potential Φ and the volume with the charge Q of the black hole. The Internal energy U is analogous to the ADM mass M of the black hole.

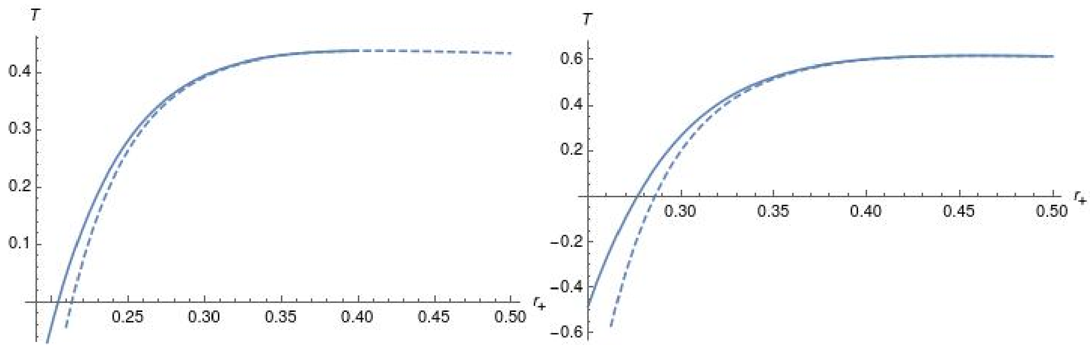
The first law can be used to calculate the black hole entropy S as

$$
S = \int_0^{r_+}\frac{1}{T}\left(\frac{\partial M}{\partial r_+}\right)_Q dr_+ = \frac{\omega}{4}r_+^{D-2}\left[1+\frac{D-2}{D-4}\frac{2\alpha'}{r_+^2}\right] \; . \qquad (2.63)
$$

The entropy computed above deviates from the standard Bekenstein relation due to the influence of the Gauss-Bonnet factor, introducing modifications that significantly impact the black hole's phase structure. Similar to the approach in the previous section, we can potentially substitute the horizon radius (r_+) in equation 2.24 with the entropy S derived from equation 2.14, thus enabling the expression of the Hawking temperature (T) in terms of the entropy (S).

Moreover, the qualitative variation of temperature with entropy can also be effectively demonstrated through temperature versus horizon radius plots, as shown in Fig.(2.10). These plots exhibit no discontinuities, signifying the absence of a first-order phase transition. To gain a deeper understanding of the behavior of charged Gauss-Bonnet black holes, we include the Maxwell limit (as $b \to \infty$) in the plots. The quantitative effects introduced by the parameter b lead to a shift in the thermodynamic temperature, indicating that a Born-Infeld black hole has a higher temperature compared to a Maxwell black hole of the same size. This effect becomes more prominent for smaller-sized black holes and diminishes as the size increases.

Furthermore, the analysis of the system's thermodynamic stability is facilitated by studying the heat capacity, which provides valuable insights into the system's stability. We proceed to analyze the system's thermodynamic stability by calculating the heat capacity at constant potential. In D dimensions, the potential takes the form

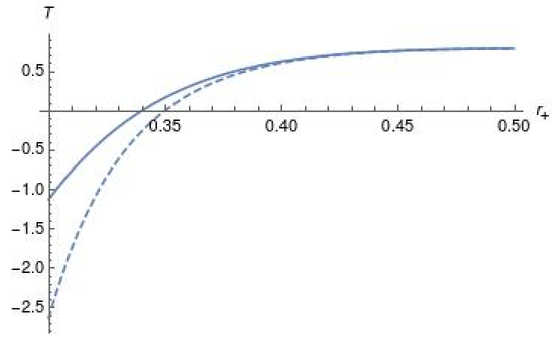


(a) D=5 (b) D=6

$$\Phi = \left(\frac{\partial M}{\partial Q}\right) = \frac{4\pi Q}{\omega(D-3) r_+^{D-3}} {}_2F_1\left[\frac{D-3}{2D-4}, \frac{1}{2}, \frac{3D-7}{2D-4}, -\frac{16\pi^2 Q^2}{b^2\omega^2 r_+^{2D-4}}\right] . \qquad (2.64)$$

The heat capacity for the black hole at constant potential can now be calculated. The general expression takes the form

$$C_\Phi \;=\; T\left(\frac{\partial S}{\partial T}\right)_\Phi$$



(a) D=7

Figure 2.10: Temperature vs Horizon Radius
Solid line(Q=0.13, b=10, α'=0.01)
Dashed line(Q=0.13, b$\rightarrow\infty$, α'=0.01)

$$= \frac{T\left(\frac{\partial\Phi}{\partial Q}\right)_{r_+}\left(\frac{\partial S}{\partial r_+}\right)_Q}{\left(\frac{\partial\Phi}{\partial Q}\right)_{r_+}\left(\frac{\partial T}{\partial r_+}\right)_Q-\left(\frac{\partial T}{\partial Q}\right)_{r_+}\left(\frac{\partial\Phi}{\partial r_+}\right)_Q}\ . \tag{2.65}$$

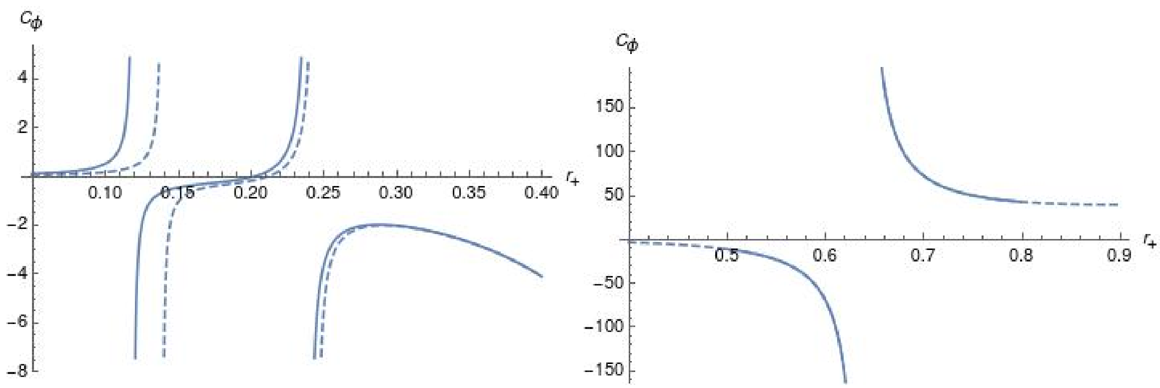
We calculate the partial derivatives in the above equation. These take the form

$$\left(\frac{\partial\Phi}{\partial r_+}\right)_Q = -\frac{4\pi Q}{\omega r_+^{D-2}}\left(1+\frac{16\pi^2Q^2}{b^2\omega^2 r_+^{2D-4}}\right)^{-1/2}\ , \tag{2.66}$$

$$\left(\frac{\partial T}{\partial Q}\right)_{r_+} = -\frac{1}{r_+^2+2\alpha'}\frac{16\pi Q}{\omega^2(D-2)r_+^{2D-7}}\left(1+\frac{16\pi^2Q^2}{b^2\omega^2 r_+^{2D-4}}\right)^{-1/2}\ , \tag{2.67}$$

$$\left(\frac{\partial S}{\partial r_+}\right)_Q \equiv \frac{dS}{dr_+} = \frac{\omega}{4}(D-2)r_+^{D-3}\left(1+\frac{2\alpha'}{r_+^2}\right)\ , \tag{2.68}$$

$$\left(\frac{\partial\Phi}{\partial Q}\right)_{r_+} = \frac{4\pi}{(D-2)\omega r_+^{D-3}}\left[\left(1+\frac{16\pi^2Q^2}{b^2\omega^2 r_+^{2D-4}}\right)^{-1/2} + \frac{1}{D-3}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{16\pi^2Q^2}{b^2\omega^2 r_+^{2D-4}}\right]\right] \tag{2.69}$$



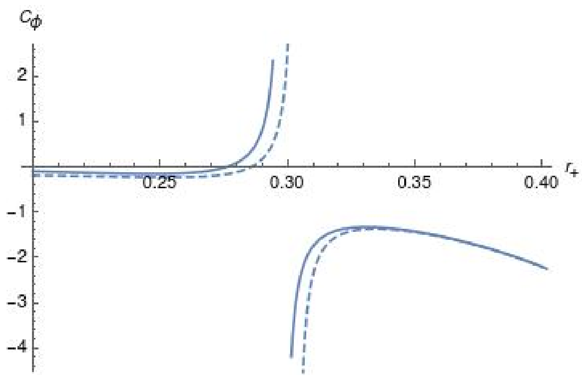
(a) D=5 (b) D=5

Figure 2.11: Heat capacity vs. Horizon Radius
Solid line(Q=0.13, b=10, α'=0.01)
Dashed line(Q=0.13, b$\rightarrow\infty$, α'=0.01)

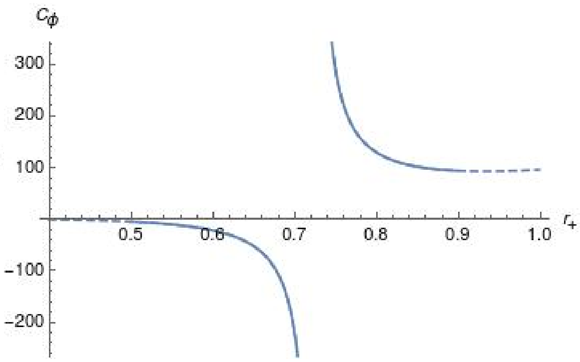
and

$$\left(\frac{\partial T}{\partial r_+}\right)_Q = \frac{1}{4\pi r_+(r_+^2+2\alpha')}\left[4(D-1)r_+^3 + 2(D-3)r_+ + \frac{16b^2r^3}{(D-2)}\left(1-\sqrt{1+\frac{16\pi^2Q^2}{b^2\omega^2 r_+^{2D-4}}}\right) + \frac{64\pi^2Q^2}{\omega^2 r_+^{2D-7}}\left(1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}\right)^{-1/2}\right]$$
$$-\frac{3r_+^2+2\alpha'}{4\pi r_+^2(r_+^2+2\alpha')^2}\left[(D-1)r_+^4+(D-3)r_+^2+(D-5)\alpha' + \frac{4b^2r_+^4}{D-2}\left(1-\sqrt{1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}}\right)\right]. \tag{2.70}$$

The heat capacity expression is too complex to present here. However, we have performed the calculations and plotted it in Fig. (s)(2.11, 2.12, 2.13) as a function of the horizon radius while keeping the other parameters fixed. The presence of multiple discontinuities in the heat capacity plots is a strong indicator of a second-order phase transition. This observation is consistent with the continuous nature of the temperature versus horizon radius plots, which, in turn, guarantees the continuity of the temperature versus entropy plots. The specific locations of these discontinuities and the radii of extremal black holes for the same parameter set are provided in Table 3.1. We denote the radius of extremal black holes as $r_+^{(e)}$ and identify the singularities in the heat capacity as $r_{+0},\ r_{+1}$, and r_{+2}.

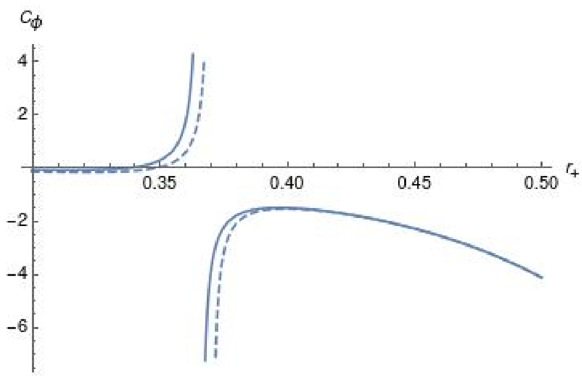


(a) D=6

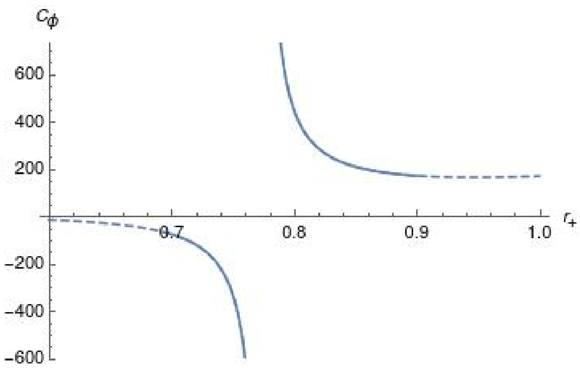


(b) D=6

Figure 2.12: Heat capacity vs. Horizon Radius
Solid line(Q=0.13, b=10, α'=0.01)
Dashed line(Q=0.13, b$\rightarrow\infty$, α'=0.01)



(a) D=7



(b) D=7

Figure 2.13: Heat capacity vs. Horizon Radius
Solid line(Q=0.13, b=10, α'=0.01)
Dashed line(Q=0.13, b$\rightarrow\infty$, α'=0.01)

	D=5	D=6	D=7
$r_+^{(e)}$	0.204957	0.276397	0.340496
r_{+0}	0.119316	–	–
r_{+1}	0.23969	0.298154	0.365921
r_{+2}	0.639584	0.725395	0.774541

Table 2.3: Extremal Radii and values of horizon radius where C_Φ diverge (For solid lines (Q=0.13, b=10, α'=0.01))

Table 3.1 reveals that the first singular point (r_{+0}) for black holes in$D = 5$ spacetime lies below the extremal black hole radius ($r_+^{(e)}$), rendering it unphysical. However, the singular points r_{+1} and r_{+2} correspond to positive temperature values. For $D = 6$ and 7, the occurrence of an unphysical divergent point does not arise. Across all spacetime dimensions, three phases are observed in the region: phase I ($r_+^{(e)} < r_+ < r_{+1}$), phase II ($r_{+1} < r_+ < r_{+2}$, and phase III ($r_+ > r_{+2}$). These phases mirror those observed in the context of black holes in Einstein's gravity. The behavior of Gauss-Bonnet-Maxwell black holes is depicted by the dotted plots. Although there are no qualitative changes, quantitatively, the Born-Infeld parameter shifts the phase transition points to lower values of the horizon radius. This effect is particularly pronounced at smaller horizon radii and diminishes as the size of the black hole increases.

We will apply the Ehrenfest scheme and Ruppeiner state space geometry techniques to determine the order of the phase transition corresponding to the points of singularities for these black holes, similar to the approach employed in the previous section. The subsequent section will begin with an exploration of the Ehrenfest scheme.

2.3.1 Ehrenfest Scheme for Phase Transition

As done for the case of the Born-Infeld black hole Einstein's gravity in the last section, we shall apply the Ehrenfest scheme for Gauss-Bonnet-Born-Infeld black holes and check the validity of the equations. We start with the left hand side of the first equation. At the points of phase transition (r_{+i}), the left hand side of the first equation can be written as

$$
\begin{aligned}
-\left[\left(\frac{\partial \Phi}{\partial T}\right)_S\right]_{S=S_i} \equiv -\left[\left(\frac{\partial \Phi}{\partial T}\right)_{r_+}\right]_{r_+=r_{+i}} &= -\left[\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+}\right]_{r_+=r_{+i}}\left[\left(\frac{\partial Q}{\partial T}\right)_{r_+}\right]_{r_+=r_{+i}} \\
&= -\frac{\left[\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+}\right]_{r_+=r_{+i}}}{\left[\left(\frac{\partial T}{\partial Q}\right)_{r_+}\right]_{r_+=r_{+i}}} . \qquad (2.71)
\end{aligned}
$$

Using eq.(s)(2.67, 2.69), the above equation takes the form

$$
\begin{aligned}
-\left[\left(\frac{\partial \Phi}{\partial T}\right)_{r_+}\right]_{r_+=r_{+i}} &= -\frac{\omega(r_{+i}^2+2\alpha')r_{+i}^{D-4}}{4Q_i}\Bigg[1 \\
&+\frac{1}{D-3}\sqrt{1+\frac{16\pi^2Q_i^2}{b^2\omega^2r_{+i}^{2D-4}}}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{16\pi^2Q_i^2}{b^2\omega^2r_{+i}^{2D-4}}\right]\Bigg] . \qquad (2.72)
\end{aligned}
$$

Also, using eq.(2.23) and the definition of heat capacity in mind $(C_\Phi = T(\frac{\partial S}{\partial T})_\Phi)$, the right hand side of eq.(2.21) becomes

$$
Q_i\beta = \left[\left(\frac{\partial Q}{\partial T}\right)_\Phi\right]_{S=S_i} = \left[\left(\frac{\partial Q}{\partial S}\right)_\Phi\right]_{S=S_i}\left(\frac{C_\Phi}{T_i}\right) \qquad (2.73)
$$

which implies

$$
\frac{\Delta C_\Phi}{T_iQ_i\Delta\beta} = \left[\left(\frac{\partial S}{\partial Q}\right)_\Phi\right]_{S=S_i} . \qquad (2.74)
$$

Using the identity (2.15), the above equation becomes

$$
\begin{aligned}
\frac{\Delta C_\Phi}{T_iQ_i\Delta\beta} &= -\frac{\left[\left(\frac{\partial \Phi}{\partial Q}\right)_S\right]_{S=S_i}}{\left[\left(\frac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i}} \\
&\equiv -\frac{\left[\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+}\right]_{r_+=r_{+i}}\left[\frac{dS}{dr_+}\right]_{r_+=r_{+i}}}{\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_Q\right]_{r_+=r_{+i}}} . \qquad (2.75)
\end{aligned}
$$

This can be calculated using eq.(s) (2.66, 2.68 , 2.69). It takes the form

$$\frac{\Delta C_\Phi}{T_i Q_i \Delta\beta} = -\frac{\omega(r_{+i}^2 + 2\alpha')r_{+i}^{D-4}}{4Q_i}\Bigg[1 + \frac{1}{D-3}\sqrt{1+\frac{16\pi^2 Q_i^2}{b^2\omega^2 r_{+i}^{2D-4}}}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{16\pi^2 Q_i^2}{b^2\omega^2 r_{+i}^{2D-4}}\right]\Bigg] \ . \tag{2.76}$$

Eq.(2.72) and eq.(2.76) are in agreement and show the validity of the first Ehrenfest equation. Next, we shall proceed to check the validity of the second Ehrenfest equation. We proceed by the same argument as in the last section. The black hole temperature can be considered a function of charge and entropy, that is, $T \equiv T(S,\Phi)$. Thus, the thermodynamic relation resulting from temperature $T(S,\Phi)$ is

$$\left(\frac{\partial T}{\partial \Phi}\right)_Q = \left(\frac{\partial T}{\partial S}\right)_\Phi \left(\frac{\partial S}{\partial \Phi}\right)_Q + \left(\frac{\partial T}{\partial \Phi}\right)_S \tag{2.77}$$

The infinite discontinuity in the heat capacity at the points of phase transition makes $\left[\left(\frac{\partial T}{\partial S}\right)_\Phi\right]_{r_+=r_{+i}} = 0$, and we know that the quantity $\left(\frac{\partial S}{\partial \Phi}\right)_Q$ is finite at the points of divergence. Therefore, the above identity at phase transition points give

$$\left[\left(\frac{\partial T}{\partial \Phi}\right)_Q\right]_{S=S_i} = \left[\left(\frac{\partial T}{\partial \Phi}\right)_S\right]_{S=S_i} \tag{2.78}$$

or

$$\left[\left(\frac{\partial T}{\partial \Phi}\right)_Q\right]_{r_+=r_{+i}} = \left[\left(\frac{\partial T}{\partial \Phi}\right)_{r_+}\right]_{r_+=r_{+i}} . \tag{2.79}$$

Thus, the left hand side of second Ehrenfest equation is equal to the left hand side of first Ehrenfest equation. So, we have

$$\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{r_+=r_{+i}} = -\frac{\omega(r_{+i}^2 + 2\alpha')r_{+i}^{D-4}}{4Q_i}\Bigg[1 + \frac{1}{D-3}\sqrt{1+\frac{16\pi^2 Q_i^2}{b^2\omega^2 r_{+i}^{2D-4}}}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{16\pi^2 Q_i^2}{b^2\omega^2 r_{+i}^{2D-4}}\right]\Bigg] \ . \tag{2.80}$$

Also, at the phase transition point, the eq.(2.23) becomes

$$\kappa Q_i = \left[\left(\frac{\partial Q}{\partial \Phi}\right)_T\right]_{S=S_i} . \tag{2.81}$$

The thermodynamic identity $\left(\frac{\partial Q}{\partial \phi}\right)_T \left(\frac{\partial \Phi}{\partial T}\right)_Q \left(\frac{\partial T}{\partial Q}\right)_\Phi = -1$, and the expression of β in eq.(2.23) lead us to

$$\kappa Q_i = \left[\left(\frac{\partial T}{\partial \Phi}\right)_Q\right]_{S=S_i} Q_i\beta . \tag{2.82}$$

Thus, the right hand side of second Ehrenfest equation (2.22) becomes

$$\frac{\Delta\beta}{\Delta\kappa} = -\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} . \tag{2.83}$$

For our convenience, this can be written in the form

$$\begin{aligned} -\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} &= -\left[\left(\frac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i}\left[\left(\frac{\partial S}{\partial T}\right)_Q\right]_{S=S_i} \\ &= -\frac{\left[\left(\frac{\partial \Phi}{\partial S}\right)_Q\right]_{S=S_i}}{\left[\left(\frac{\partial T}{\partial S}\right)_Q\right]_{S=S_i}} \end{aligned} \tag{2.84}$$

which simplifies to

$$-\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{S=S_i} \equiv -\left[\left(\frac{\partial \Phi}{\partial T}\right)_Q\right]_{r_+=r_{+i}} = -\frac{\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_Q\right]_{r_+=r_{+i}}}{\left[\left(\frac{\partial T}{\partial r_+}\right)_Q\right]_{r_+=r_{+i}}} . \tag{2.85}$$

The divergence of heat capacity in eq.(2.65) gives the condition

$$\left[\left(\frac{\partial \Phi}{\partial Q}\right)_{r_+}\right]_{r_+=r_{+i}}\left[\left(\frac{\partial T}{\partial r_+}\right)_Q\right]_{r_+=r_{+i}} = \left[\left(\frac{\partial T}{\partial Q}\right)_{r_+}\right]_{r_+=r_{+i}}\left[\left(\frac{\partial \Phi}{\partial r_+}\right)_Q\right]_{r_+=r_{+i}} . \tag{2.86}$$

The above condition and eq.(2.85) lead to the form of right hand side of the second Ehrenfest equation to be

$$\frac{\Delta\beta}{\Delta\kappa}=-\frac{\omega(r_{+i}^{2}+2\alpha')r_{+i}^{D-4}}{4Q_i}\left[1\right.$$
$$\left.+\frac{1}{D-3}\sqrt{1+\frac{16\pi^2Q_i^2}{b^2\omega^2r_{+i}^{2D-4}}}\,{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{16\pi^2Q_i^2}{b^2\omega^2r_{+i}^{2D-4}}\right]\right]\ . \tag{2.87}$$

Thus, Eq.(s)(2.80,2.87) show the validity of the second Ehrenfest equation at the phase transition point. The validity of both equations confirm the second order nature of phase transition points in the considered black hole system. In the subsequent section, we shall carry out geometric analysis of the phase transition.

2.3.2 Ruppeiner State Space Geometry and Phase Transition

Next, we computer the Ruppeiner's metric coefficients and scalar curvature in general D spacetime dimensions. The metric coefficients for the system take the form

$$\begin{aligned}
g_{SS} &= \frac{1}{T}\left[\frac{\left(\frac{\partial^2 M}{\partial r_+^2}\right)_Q}{\left(\frac{dS}{dr_+}\right)^2}-\frac{\left(\frac{\partial M}{\partial r_+}\right)_Q\left(\frac{d^2S}{dr_+^2}\right)}{\left(\frac{dS}{dr_+}\right)^3}\right]\\
g_{SQ} &= \frac{1}{T}\left[\frac{\frac{\partial^2 M}{\partial r_+\partial Q}}{\frac{dS}{dr_+}}\right]\\
g_{QQ} &= \frac{1}{T}\left(\frac{\partial^2 M}{\partial Q^2}\right)_{r_+}\ .
\end{aligned} \tag{2.88}$$

The partial differentials involved in the above equations are calculated from eq.(s)(2.59, 2.63). These take the form

$$\frac{\partial^2 M}{\partial r_+\partial Q}=-\frac{4\pi Q}{\omega r_+^{D-2}}\left(1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}\right)^{-1/2} \tag{2.89}$$

$$\frac{d^2S}{dr_+^2}=\frac{\omega}{4}(D-2)r_+^{D-4}\left((D-3)+\frac{2(D-5)\alpha'}{r_+^2}\right) \tag{2.90}$$

$$\left(\frac{\partial M}{\partial r_+}\right)_Q = \frac{(D-2)(D-5)\omega\alpha'}{16\pi}r_+^{D-6} + \frac{(D-2)(D-3)\omega}{16\pi}r_+^{D-4}$$
$$+\frac{(D-1)(D-2)\omega)}{16\pi}r_+^{D-2} + \frac{b^2\omega}{4\pi}r_+^{D-2}\left(1-\sqrt{1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}}\right) \tag{2.91}$$

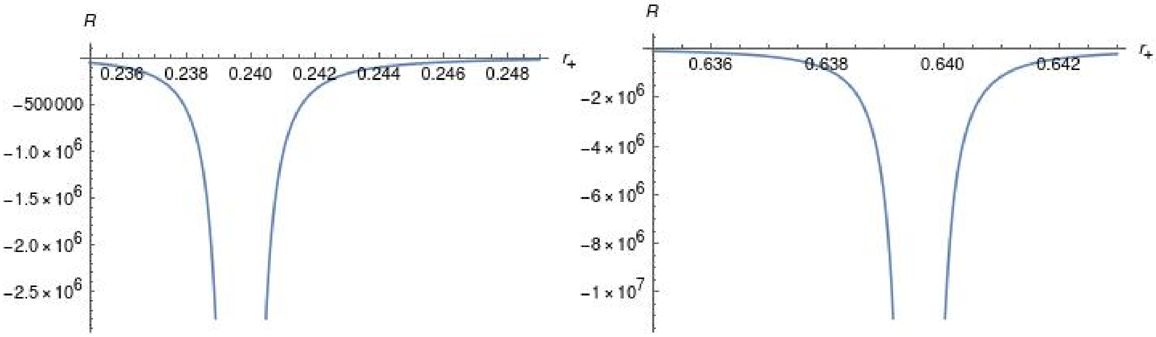
$$\left(\frac{\partial^2 M}{\partial r_+^2}\right)_Q = \frac{(D-2)(D-5)(D-6)\omega\alpha'}{16\pi}r_+^{D-7} + \frac{(D-2)(D-3)(D-4)\omega}{16\pi}r_+^{D-5}$$
$$+\frac{(D-1)(D-2)^2\omega)}{16\pi}r_+^{D-3} + \frac{b^2(D-2)\omega}{4\pi}r_+^{D-3}\left(1-\sqrt{1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}}\right)$$
$$+\frac{4(D-2)\pi Q^2}{\omega r_+^{D-1}}\left(1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}\right)^{-1/2} \tag{2.92}$$

$$\left(\frac{\partial^2 M}{\partial Q^2}\right)_{r_+} = \frac{4\pi}{\omega(D-2)r_+^{D-3}}\left[\left(1+\frac{16\pi^2Q^2}{b^2\omega^2r_+^{2D-4}}\right)^{-1/2}\right.$$
$$\left.+\frac{1}{D-3}{}_2F_1\left[\frac{D-3}{2D-4},\frac{1}{2},\frac{3D-7}{2D-4},-\frac{16\pi^2Q_i^2}{b^2\omega^2r_{+i}^{2D-4}}\right]\right] . \tag{2.93}$$

When considering the partial derivatives with respect to the horizon radius, the definition of Ruppeiner curvature in Eq. (2.51) is modified to the following form

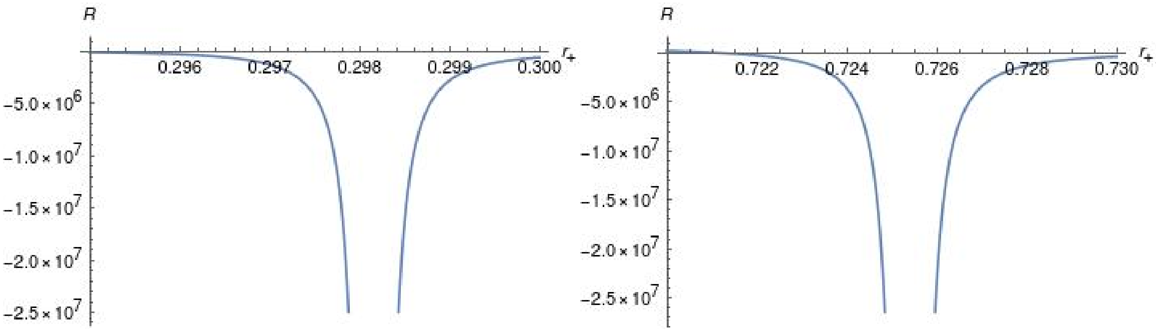
$$R = -\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial r_+}\left(\frac{g_{SQ}}{\sqrt{g}g_{SS}}\frac{\partial g_{SS}}{\partial Q} - \frac{1}{\sqrt{g}}\frac{g_{QQ}}{\partial r_+}\left(\frac{dr_+}{dS}\right)\right)\left(\frac{dr_+}{dS}\right)\right.$$
$$\left.+\frac{\partial}{\partial Q}\left(\frac{2}{\sqrt{g}}\frac{\partial g_{SQ}}{\partial Q} - \frac{1}{\sqrt{g}}\frac{\partial g_{SS}}{\partial Q} - \frac{g_{SQ}}{\sqrt{g}g_{SS}}\frac{\partial g_{SS}}{\partial r_+}\left(\frac{dr_+}{dS}\right)\right)\right] \tag{2.94}$$

where g is $g_{SS}\times g_{QQ}-g_{SQ}^2$. We plotted R with the horizon radius (r_+) for the same values of other parameters in fig.(s)(2.14,2.15, 2.16). We plotted in the range where first and second phase transition points are visible separately. The infinite discontinuity occur for those points only where the heat capacity diverged. After the validity of Ehrenfest scheme, this reconfirms the second order nature of phase transition for Gauss-Bonnet-Born-Infeld black holes as well.



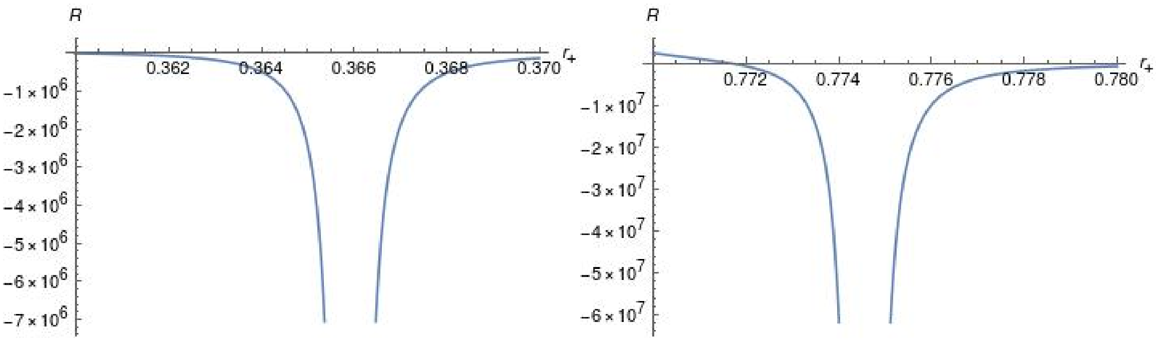
(a) D=5, Q=0.13, b=10, α'=0.01, Critical point (r_{+1}) (b) D=5, Q=0.13, b=10, α'=0.01, Critical point (r_{+2})

Figure 2.14: Ruppeiner Curvature vs Horizon Radius



(a) D=6, Q=0.13, b=10, α'=0.01, Critical point (r_{+1}) (b) D=6, Q=0.13, b=10, α'=0.01, Critical point (r_{+2})

Figure 2.15: Ruppeiner Curvature vs Horizon Radius



(a) D=7, Q=0.13, b=10, α'=0.01, Critical point (r_{+1}) (b) D=7, Q=0.13, b=10, α'=0.01, Critical point (r_{+2})

Figure 2.16: Ruppeiner Curvature vs Horizon Radius

2.4 Conclusion and Remarks

In this chapter of the thesis, we conducted a comprehensive thermodynamic analysis of Born-Infeld black holes in both Einstein gravity and Gauss-Bonnet gravity within an constant AdS background. Our examination encompassed the investigation of phase transition properties within a constant AdS radius across general D spacetime dimensions. The analysis involved a thorough examination of black holes across the entire range of horizon radii, with a particular focus on calculating the heat capacity at constant electric potential.

The heat capacity plots revealed a finite number of infinite discontinuities. Notably, for black holes in both Einstein gravity and Gauss-Bonnet gravity, the plots of temperature versus entropy remained continuous, indicating the absence of a first-order phase transition. To further explore these discontinuities in heat capacity, we employed two well-known techniques from standard thermodynamics: the Ehrenfest scheme of phase transition and Ruppeiner state space geometry analysis.

Our analysis characterized the phase transition points as second-order critical points, with the black holes existing in three distinct phases based on the sign of the heat capacity: the small stable black hole phase, intermediate unstable phase, and large stable black hole phase. We found that the behavior closely resembled that of Reissner-Nordstrom black holes. Moreover, we extensively discussed the quantitative dependence on the Gauss-Bonnet and Born-Infeld parameters.

The system successfully satisfied the Ehrenfest equations for the second-order phase transition. The singularities in Ruppeiner curvature corresponded to those values of the horizon radius where the heat capacity diverged, confirming the second-order nature of the phase transition. However, since infinite discontinuities in heat capacity are not commonly observed in standard thermodynamic systems, drawing conclusions about the microscopic understanding of the phase transition remains challenging. Thus, the exploration of a microscopic theory becomes imperative to provide insights into the qualitative changes during the second-order phase transition.

CHAPTER 3

Black Hole Phase Transition in Extended Phase Space

3.1 Introduction

The extension of the thermodynamic phase space for black holes is imperative, especially considering the inconsistencies in the Smarr relation [42], which have been extensively discussed in Chapter 1. This expansion of the thermodynamic phase space has significantly enhanced our comprehension of phase transitions in black holes. In addition to showcasing van der Waals fluid-like behavior, black holes have also been observed to undergo superfluid [126], re-entrant [127], and polymer-type [128] phase transitions.

The extension of the phase space has also sparked considerable interest from the perspective of gauge/gravity duality. This duality is notably recognized as the AdS/CFT conjecture in the presence of a negative cosmological constant. According to this conjecture, a black hole's theory in AdS space is dually connected to the conformal field theory at its asymptotic boundary. Both theories are linked by an equality between the gravity and field theory partition functions, famously termed the GKPW relation [101]. Consequently, the thermodynamic variables in the bulk and the boundary are believed to be dual to each other, implying a correspondence between the first laws of the two theories. On account of the extension of thermodynamic phase space, a pressure-volume term has been included in the first law. This allows black holes to be conceptualized as heat engines, extensively investigated in [129–137], with the primary aim of comprehending the holographic dual depiction of processes within the bulk. However, this correspondence has revealed certain ambiguities [138–142]. These ambiguities can be comprehended through the following argument: the variation of Λ in the bulk is correlated with a variation in the central charge of the boundary field theory [145]. In other words, a modification in Λ alters the theory space at the boundary.

Furthermore, another complication arises with the variation of Λ in the bulk, affecting the one-to-one correspondence with the boundary variables. A change in the AdS radius also corresponds to a change in the boundary volume [129], indicating that alterations in the system size at the boundary are inevitable. Therefore, the treatment of the cosmological constant as a thermodynamic pressure poses significant conceptual challenges within the context of the holographic dual theory.

In a recent paper by Mann et al. [146], an innovative approach was proposed, suggesting the variation of Newton's constant in the bulk, thus treating it as a thermodynamic variable. This approach facilitates the alteration of pressure and Newton's constant in the bulk without affecting the central charge for a system at the boundary. From the viewpoint of the bulk, this technique allows the expression of the first law in terms of boundary variables alongside other essential thermodynamic parameters of the black hole. Referred to as the 'mixed first law', this form enables the study of the phase structure in a scenario where the central charge is considered as another parameter in addition to Λ.

The authors of [146] have also identified a new variable that is conjugate to pressure (i.e., Λ), distinct from the volume in extended black hole thermodynamics, adding a new understanding to the study. Furthermore, an intriguing observation has been made concerning the phase transition properties of the black hole. Although van der Waals fluid-like behavior persists, the critical value of the central charge, denoting the value of the parameter at the point of inflection, is deemed 'universal'. In this context, 'universality' refers to the independence of this critical value from other thermodynamic parameters.

Expanding our this study, we included an analysis of Born-Infeld and Gauss-Bonnet black holes. Our objective is to explore the potential alterations these modifications might bring to the phase transition properties, in addition to investigating their impact on the feature of universality. The contents of this chapter are derived from the following two papers:

- **Neeraj Kumar**, Soham Sen, and Sunandan Gangopadhyay, *Breaking of the universal* *nature of the central charge criticality in AdS black holes in Gauss-Bonnet gravity*, Phys. Rev. D 107, 046005 – Published 13 February 2023

- **Neeraj Kumar**, Soham Sen, and Sunandan Gangopadhyay, *Phase transition structure and breaking of universal nature of central charge criticality in a Born-Infeld AdS black hole*, Phys. Rev. D 106, 026005 – Published 18 July 2022

We shall start with the Born-Infeld black holes first and discuss Gauss-Bonnet black holes later.

3.2 Born-Infeld AdS Black Hole in Extended Phase Space

3.2.1 Hawking Temperature and Thermodynamics

Thermodynamic properties of a Born-Infeld AdS black hole shall be reviewed here. The action in $(3+1)$ dimensions has the form

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - 2\Lambda + L(F)] \tag{3.1}$$

where the Born-Infeld Lagrangian density is of the form

$$L(F) = 4b^2\left(1 - \sqrt{1 + \frac{F^{\mu\nu}F_{\mu\nu}}{2b^2}}\right)\ . \tag{3.2}$$

Here, Λ is the cosmological constant and b is the Born-Infeld parameter. G is retained in the calculations as we shall treat it as a thermodynamic variable later. Here, we will work in units $\hbar = c = k_B = 1$.

A static, spherically symmetric solution with Λ was first discovered in [81]. Under these ansatz the black hole solution for the above action has the form [81]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2\ , \tag{3.3}$$

where

$$f(r) = 1 - \frac{2GM}{r} + \frac{r^2}{l^2} + \frac{2b^2r^2}{3}\left(1 - \sqrt{1 + \frac{GQ^2}{b^2r^4}}\right) + \frac{4GQ^2}{3r^2}\ {}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{GQ^2}{b^2r^4}\right]\ . \tag{3.4}$$

Here, M is the black hole mass, Q is the charge associated with it, l is the AdS radius parameter. ${}_2F_1$ is a hypergeometric function of the form ${}_2F_1[a,b,c,z]$. The Reissner-Nordstörm AdS black hole solution recovers in the limit $b \to \infty$. The cosmological constant Λ and AdS radius l are related as $\Lambda = -\frac{3}{l^2}$ in $(3+1)$-dimensions.

The root of equation $f(r_+) = 0$ will give the expression of the event horizon of the black hole. Also, the mass M of the black hole can be expressed in terms of horizon radius (r_+) as

$$M = \frac{r_+}{2G} + \frac{r_+^3}{2Gl^2} + \frac{b^2 r_+^3}{3G}\left(1 - \sqrt{1 + \frac{GQ^2}{b^2 r_+^4}}\right) + \frac{2Q^2}{3r_+}\,{}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{GQ^2}{b^2 r_+^4}\right] . \tag{3.5}$$

The Hawking temperature of the black hole can be obtained from eq.(3.4) as follows

$$T = \frac{1}{4\pi}\frac{\partial f}{\partial r}\bigg|_{r=r_+} = \frac{1}{4\pi}\left[\frac{1}{r_+} + \frac{3r_+}{l^2} + 2b^2 r_+\left(1 - \sqrt{1 + \frac{GQ^2}{b^2 r_+^4}}\right)\right] . \tag{3.6}$$

These are the thermodynamic quantities associated to Born-Infeld black holes. In the forthcoming section, the modified form of the first law of black hole thermodynamics is derived in D spacetime dimensions.

3.2.2 Modified First Law

As emphasized in the introductory chapter, the incorporation of dimensionful parameters as thermodynamic variables is crucial to maintain the consistency of the Smarr relation. In the case of Born-Infeld black holes, the inclusion of the Born-Infeld parameter in the first law becomes imperative, given its dimensional nature $[b] = L^1$. This approach has also been recommended in [93].

Also, pressure in terms of the cosmological constant and AdS radius in general D spacetime dimensions is given by [92]

$$P = -\frac{\Lambda}{8\pi G} = -\frac{(D-1)(D-2)}{2l^2} . \tag{3.7}$$

It is defined such that the negative cosmological constant gives positive thermodynamic pressure.

The Bekenstein-Hawking entropy formula and the Hawking temperature in natural units take the form

$$S = \frac{A}{4G} \qquad T = \frac{\kappa}{2\pi} \ . \tag{3.8}$$

Now equipped with all the essential variables, we proceed to establish the first law in the extended phase space. As previously emphasized, the mass of the black hole represents the total enthalpy of the black hole within the extended thermodynamic phase space [92], rather than its internal energy. Consequently, the first law of thermodynamics for the most general Born-Infeld black holes (with all possible thermodynamic charges) takes the following form [92–94]

$$\begin{aligned} \delta M &= T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J + \mathcal{B}\delta b \\ &= \frac{\kappa}{2\pi}\delta S + V\delta P + \Phi\delta Q + \Omega\delta J + \mathcal{B}\delta b \\ &= \frac{\kappa}{8\pi G}\delta A - \frac{V}{8\pi G}\delta\Lambda + \Phi\delta Q + \Omega\delta J + \mathcal{B}\delta b \ . \end{aligned} \tag{3.9}$$

The physical significance of variable conjugate to the Born-Infeld parameter b has aleady been discussed in details in [94]. The variable has been termed as *Born-Infeld vacuum polarization*. As highlighted in the introduction, there exists an inconsistency in the holographic interpretation of the aforementioned first law. Specifically, variations in the cosmological constant within the bulk result in modifications to both the central charge of the dual theory at the boundary and the volume of the system under consideration at the boundary. In a step towards understanding the resolution [146], we start with utilizing the AdS/CFT dictionary, specifically, a relationship between the bulk parameters and the central charge at the boundary, which is of the form [145].

$$C = k\frac{l^{D-2}}{16\pi G} \ , \tag{3.10}$$

where the factor k has the information of the particular system considered at the boundary. From the perspective of boundary system, it is worthwhile to consider a system which can be explained by one theory (that is, when the central charge remains fixed). It is clear from the above expression that in order to do so, we need to vary the Newton's constant G along with l in the bulk such that C remains fixed. From the bulk perspective, we now have a thermodynamic system where the Newton's constant is a themrodynamic variable too. In case of Born-Infeld black holes, we already know that the Born-Infeld parameter being dimensionful appears in the

Smarr formula and first law [94]. Thus, we can consider mass of the black hole as

$$M \equiv M(A, Q, J, G, \Lambda, b)\ . \tag{3.11}$$

The mass variation in terms of these variables shall take the form

$$\delta M = \frac{\partial M}{\partial A}\delta A + \frac{\partial M}{\partial Q}\delta Q + \frac{\partial M}{\partial J}\delta J + \frac{\partial M}{\partial G}\delta G + \frac{\partial M}{\partial \Lambda}\delta\Lambda + \frac{\partial M}{\partial b}\delta b. \tag{3.12}$$

We now define $G\frac{\partial M}{\partial G} = -\xi$ for ease in calculations. Comparing eq.(3.12) with eq.(3.9), the conjugate variables of $A,\ \Lambda,\ Q, b$ and J are identified as $\frac{\kappa}{8\pi G}$, $-\frac{V}{8\pi G}$, Φ, $\mathcal{B}$ and Ω respectively. Thus, we can write eq.(3.12) as

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Phi\delta Q + \Omega\delta J - \xi\frac{\delta G}{G} - \frac{V}{8\pi G}\delta\Lambda + \mathcal{B}\delta b\ . \tag{3.13}$$

We shall want to compute the coefficient ξ of δG. For that, we follow the technique suggested in [146] and write a modified mass term as

$$GM = \mathcal{M}(A, \sqrt{G}Q, \Lambda, GJ, b)\ . \tag{3.14}$$

Differential of the above relation give us

$$\begin{aligned}
\delta(GM) &= \frac{\partial\mathcal{M}}{\partial A}\delta A + \frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)}\delta(\sqrt{G}Q) + \frac{\partial\mathcal{M}}{\partial\Lambda}\delta\Lambda + \frac{\partial\mathcal{M}}{\partial(GJ)}\delta(GJ) + \frac{\delta\mathcal{M}}{\delta b}\delta b \\
\implies G\delta M &= -M\delta G + \frac{\partial\mathcal{M}}{\partial A}\delta A + \sqrt{G}\frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)}\delta Q + \frac{Q}{2\sqrt{G}}\frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)}\delta G + J\frac{\partial\mathcal{M}}{\partial(GJ)}\delta G \\
&\quad + G\frac{\partial\mathcal{M}}{\partial(GJ)}\delta J + \frac{\partial\mathcal{M}}{\partial\Lambda}\delta\Lambda + \frac{\partial\mathcal{M}}{\partial b}\delta b \\
\implies \delta M &= \frac{1}{G}\frac{\partial\mathcal{M}}{\partial A}\delta A + \frac{1}{\sqrt{G}}\frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)}\delta Q + \frac{1}{G}\left(-M + \frac{Q}{2\sqrt{G}}\frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)} + J\frac{\partial\mathcal{M}}{\partial(GJ)}\right)\delta G \\
&\quad + \frac{\partial\mathcal{M}}{\partial\Lambda}\delta\Lambda + \frac{\partial\mathcal{M}}{\partial(GJ)}\delta J + \frac{\partial\mathcal{M}}{\partial b}\delta b\ .
\end{aligned} \tag{3.15}$$

Comparison of eq.(3.15) with eq.(3.13) gives us

$$\frac{\partial \mathcal{M}}{\partial A} = \frac{\kappa}{8\pi} \; , \; \frac{1}{\sqrt{G}} \frac{\partial \mathcal{M}}{\partial(\sqrt{G}Q)} = \Phi \; , \; \frac{\partial \mathcal{M}}{\partial(GJ)} = \Omega \; , \frac{\partial \mathcal{M}}{\partial \Lambda} = -\frac{V}{8\pi G} \; , \; \frac{\partial \mathcal{M}}{\partial b} = \mathcal{B} \tag{3.16}$$

and the variable conjugate to G takes the form

$$\xi = M - \frac{Q}{2\sqrt{G}} \frac{\partial \mathcal{M}}{\partial(\sqrt{G}Q)} - J \frac{\partial \mathcal{M}}{\partial(GJ)} \; . \tag{3.17}$$

This can be written in the form of other black hole parameters as

$$\xi = M - \frac{Q\Phi}{2} - \Omega J \; . \tag{3.18}$$

This required form of the conjugate variable is obtained using eq.(3.16) in eq.(3.17).

Now we proceed to derive a mixed form of first law. The differential of eq.(3.10), when divided by C takes the form

$$\frac{\delta C}{C} = -\frac{\delta G}{G} + (D-2)\frac{\delta l}{l} \tag{3.19}$$

which further simplifies to

$$\frac{\delta l}{l} = -\frac{\delta G}{2G} - \frac{\delta P}{2P}. \tag{3.20}$$

Eq.(3.20) enables us to recast eq.(3.19) in the form

$$\frac{\delta G}{G} = -\frac{2}{D}\frac{\delta C}{C} - \frac{(D-2)}{D}\frac{\delta P}{P} \; . \tag{3.21}$$

Thus, eq.(3.15) can now be written in the form

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Phi \delta Q + \Omega \delta J - \frac{V}{8\pi G}\delta\Lambda + \mathcal{B}\delta b + \frac{2\xi}{DC}\delta C + \frac{(D-2)}{D}\xi\frac{\delta P}{P} \; . \tag{3.22}$$

Using eq.(s)(3.7, 3.8, 3.21), we can rewrite eq.(3.22) as

$$\begin{aligned} \delta M &= T\delta S + \phi\delta Q + \Omega\delta J + \left[\frac{2\xi}{DC} - \frac{2(TS+PV)}{DC}\right]\delta C \\ &\quad + \left[V + \frac{D-2}{DP}\xi - \frac{D-2}{DP}(TS+PV)\right]\delta P + \mathcal{B}\delta b \\ &= T\delta S + \phi\delta Q + \Omega\delta J + \mathcal{B}\delta b + V_{\mathcal{C}}\delta P + \mu_{\mathcal{C}}\delta C \end{aligned} \tag{3.23}$$

where

$$V_{\mathcal{C}} = V + \frac{D-2}{DP}\xi - \frac{D-2}{DP}(TS+PV), \quad \mu_{\mathcal{C}} = \frac{2\xi}{DC} - \frac{2(TS+PV)}{DC} \,. \tag{3.24}$$

These variables $V_{\mathcal{C}}$ and $\mu_{\mathcal{C}}$ are conjugate to thermodynamic volume and chemical potential. Eq.(3.23) is called the mixed form of the first law of thermodynamics when Born-Infeld electromagnetic fields are considered in AdS black holes. This allows us to study thermodynamics of the black hole while keeping the central charge fixed, even when G and Λ are varied. The name, mixed first law, is because it contains bulk and boundary variables both.

3.2.3 Smarr Relation in Extended Black Hole Thermodynamics

This subsection is devoted to the calculation of V in terms of black hole parameters such as the mass and charge of the black hole. This is done by deriving the Smarr relation using dimensional analysis, as suggested in the introductory chapter.

The first law for Born-Infeld AdS black holes can be written as [94]

$$\delta M = T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J + \mathcal{B}\delta b \,. \tag{3.25}$$

Here, Φ, Ω and $\mathcal{B}$ are variables conjugate to Q, J and b respectively, where

$$\Phi = \frac{\delta M}{\delta Q} \,, \quad \Omega = \frac{\delta M}{\delta J} \,, \quad \mathcal{B} = \frac{\delta M}{\delta b} \,. \tag{3.26}$$

The ADM mass can be considered as a function of $M \equiv M(S, P, b, Q, J)$. In general D-dimensions, M, S, P, b, Q, J have the following dimensions (in terms of length L)

$$\begin{aligned} &[M] = L^{D-3}, \; [S] = L^{D-2}, \; [P] = L^{-2} \,, \\ &[b] = L^{-1}, \; [Q] = L^{D-3}, \; [J] = L^{D-2} \,. \end{aligned} \tag{3.27}$$

Utilising Euler's theorem of quasi-homogeneous functions, we have

$$\begin{aligned}(D-3)M &= (D-2)S\frac{\delta M}{\delta S} - b\frac{\delta M}{\delta b} - 2P\frac{\delta M}{\delta P} \\ &+ (D-3)Q\frac{\delta M}{\delta Q} + (D-2)J\frac{\delta M}{\delta J} \,.\end{aligned} \tag{3.28}$$

From the first law in eq.(3.25), we can write $\frac{\partial M}{\partial S} = T$, $\frac{\partial M}{\partial Q} = \Phi$, $\frac{\partial M}{\partial P} = V$, $\frac{\delta M}{\delta J} = \Omega$ and $\frac{\partial M}{\partial b} = \mathcal{B}$. Substituting these in eq.(3.28), we obtain the Smarr formula which has the following form

$$(D-3)M = (D-2)TS - \mathcal{B}b - 2PV + (D-3)\Phi Q + (D-2)\Omega J \,. \tag{3.29}$$

From eq.(3.29), we can write the required form of black hole volume V in terms of other black hole parameters as

$$\begin{aligned}V &= \frac{D-2}{2P}TS - \frac{\mathcal{B}b}{2P} + \frac{D-3}{2P}\Phi Q - \frac{D-3}{2P}M + \frac{D-2}{2P}\Omega J \\ &= \frac{D-3}{2P}\left(\frac{D-2}{D-3}(TS + \Omega J) + \Phi Q - M - \frac{1}{D-3}\mathcal{B}b\right) \,.\end{aligned} \tag{3.30}$$

3.2.4 Modified Thermodynamic Variables

The expression of V from eq.(3.30), when put back in eq.(3.24), becomes

$$V_{\mathcal{C}} = \frac{2M - 2\mathcal{B}b + (D-4)Q\Phi}{2DP} \,. \tag{3.31}$$

For $D = 4$, eq.(3.31) reduces to the form

$$V_{\mathcal{C}} = \frac{M - \mathcal{B}b}{DP} \tag{3.32}$$

or

$$V_{\mathcal{C}} = \frac{4\pi r_+^3}{3} + \frac{4\pi l^2 r_+}{3} - \frac{8\pi b^2 l^2 r_+^3}{9} + \frac{8\pi b^2 l^2 r_+^3}{9}\sqrt{1 + \frac{GQ^2}{b^2 r_+^4}} + \frac{8\pi G Q^2 l^2}{9 r_+}\, {}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{GQ^2}{b^2 r_+^4}\right] , \tag{3.33}$$

where we used eq.(3.5), and the following equations

$$P = \frac{3}{8\pi l^2 G}, \; C = \frac{kl^2}{16\pi G}, \; \Phi = \frac{Q}{r_+} \, {}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{GQ^2}{b^2 r_+^4}\right] , \tag{3.34}$$

and

$$\mathcal{B} = \frac{\partial M}{\partial b} = \frac{2br_+^3}{3G}\left[1 - \sqrt{1 + \frac{GQ^2}{b^2 r_+^4}}\right] + \frac{Q^2}{3br_+} \, {}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{GQ^2}{b^2 r_+^4}\right] \tag{3.35}$$

to obtain the form of $V_{\mathcal{C}}$. The new thermodynamic variable $V_{\mathcal{C}}$ can be explicitly seen to depend on the Born-Infeld parameter, signifying the importance of the Born-Infeld parameter in the novel definition of volume parameter.

Next, we are interested in understanding the effects of the Born-Infeld term on the critical behaviour and possible phase transition properties of the black hole.

3.2.5 Central Charge Universality Breaking

Reissner-Nordstrom AdS black holes show a small-large phase transition with characteristics similar to van der Waals fluid when the mixed first is considered [146]. It is straight forward to expect this in case of Born-Infeld AdS black holes as well. Here, we start with calculating the critical value of the central charge first. The critical point corresponds to the solution of the following two equations [146]

$$\frac{\partial T}{\partial r_+} = 0 \, , \tag{3.36}$$

$$\frac{\partial^2 T}{\partial r_+^2} = 0 \, . \tag{3.37}$$

The expression of Hawking temperature in eq.(3.6), when differentiated twice, gives us

$$1 - \frac{6GQ^2}{r_{+(c)}^2\left(1 + \frac{GQ^2}{b^2 r_{+(c)}^4}\right)^{\frac{1}{2}}} + \frac{4G^2Q^4}{b^2 r_{+(c)}^4\left(1 + \frac{GQ^2}{b^2 r_{+(c)}^4}\right)^{\frac{3}{2}}} = 0 \, . \tag{3.38}$$

Here, the subscript c denotes the critical value. A complete solution of the above equation is difficult to obtain. We approximate eq.(3.38) to an order of $\mathcal{O}(1/b^2)$, and it takes the form

$$r_{+(c)}^6 - 6GQ^2 r_{+(c)}^4 + \frac{7G^2Q^4}{b^2} = 0 \, . \tag{3.39}$$

The solution of eq.(3.39) can only be obtained by perturbative approach. We assume a solution of the form for $r_{+(c)}$ up to $\mathcal{O}(1/b^2)$ as follows

$$r_{+(c)} \cong r_+^{(0)} + \frac{r_+^{(1)}}{b^2} \ . \tag{3.40}$$

Using eq.(3.40) in eq.(3.39), the forms of solutions of $r_+^{(0)}$ and $r_+^{(1)}$ are

$$r_+^{(0)} = \sqrt{6G}Q \ , \quad r_+^{(1)} = -\frac{7}{72\sqrt{6G}Q} \ . \tag{3.41}$$

The above equation provides the form of the critical value of $r_{+(c)}$ up to $\mathcal{O}(1/b^2)$ as

$$r_{+(c)} \cong \sqrt{6G}Q - \frac{7}{72\sqrt{6G}Qb^2} \ . \tag{3.42}$$

Also, eq.(3.36) can be re-written in the following form

$$-\frac{1}{r_{+(c)}^2} + \frac{3}{l^2} + 2b^2 - 2b^2\sqrt{1 + \frac{GQ^2}{b^2 r_{+(c)}^4}} + \frac{4GQ^2}{r_{+(c)}^4\sqrt{1 + \frac{GQ^2}{b^2 r_{+(c)}^4}}} = 0 \ . \tag{3.43}$$

The above equation up to $\mathcal{O}(1/b^2)$ takes the following form

$$-\frac{1}{r_{+(c)}^2} + \frac{3}{l^2} - \frac{7G^2Q^4}{4b^2 r_{+(c)}^8} + \frac{3GQ^2}{r_{+(c)}^4} = 0 \ . \tag{3.44}$$

By substituting eq.(3.42) in eq.(3.44), we obtain the critical value of the AdS radius up to $\mathcal{O}(1/b^2)$ as

$$l_c \cong 6\sqrt{G}Q\left(1 - \frac{7}{864GQ^2b^2}\right) \ . \tag{3.45}$$

Substituting the values of $r_{+(c)}$ and l_c in eq.(3.6), we get the critical value of the Hawking temperature up to $\mathcal{O}(1/b^2)$ as

$$T_c = \frac{1}{3\sqrt{6G}\pi Q} + \frac{1}{432\pi\sqrt{6G}GQ^3b^2} \ . \tag{3.46}$$

The results reduce to the Reissner-Nordström black hole case in the limit ($b \to \infty$) [146]. The value of the critical charge can be obtained by using the form of l_C from eq.(3.45) in eq.(3.10) up to $\mathcal{O}(1/b^2)$ as follows

$$\begin{aligned} C_c &= k\frac{l_c^2}{16\pi G} \\ \implies C_c &\cong \frac{9kQ^2}{4\pi} - \frac{7k}{16\pi G b^2} \ . \end{aligned} \tag{3.47}$$

The above equation gives the form of the critical central charge in terms of $Q,\ G$ and b. It can also be expressed in terms of $Q,\ b$ and the critical pressure P_c. This helps us to avoid using values of G when calculating the values of the critical central charge C_c for different values of the parameter b (Table 3.1).

For $D = 4$ and for $l = l_c$, eq.(3.7) becomes

$$P_c = \frac{3}{8\pi G l_c^2} \ . \tag{3.48}$$

Replacing l_c^2 by $\frac{16\pi G C_c}{k}$ in the above equation, we obtain

$$G = \sqrt{\frac{3k}{128\pi^2 C_c P_c}} \tag{3.49}$$

where P_c is the critical value of pressure. Replacing G from eq.(3.47), we can write eq.(3.47) in the form

$$C_c = \frac{9kQ^2}{4\pi} - \frac{7\sqrt{kP_c}}{\sqrt{6}b^2} C_c^{\frac{1}{2}} \ . \tag{3.50}$$

The above equation, when solved up to $\mathcal{O}(1/b^2)$, the critical charge in terms of P_c, the Born-Infeld parameter b, charge Q and k becomes

$$C_c \cong \frac{9kQ^2}{4\pi} - \frac{7kQ}{4b^2}\sqrt{\frac{6P_c}{\pi}} \ . \tag{3.51}$$

For Reissner-Nordström black hole, the critical value of the critical central charge as obtained in [146] is

$$C_c = \frac{9kQ^2}{4\pi} \ . \tag{3.52}$$

As discussed in [146], the critical value of central charge C_c is universal in the sense that it depends only on Q but not other parameters. However, this changes for Born-Infeld black holes in AdS background. From eq.(3.51), it is clear that the universal nature breaks down and

Table 3.1: Dependence of C_c on b

b	10	15	20	30	100	∞
C_c	31.29	33.91	34.82	35.48	35.95	36

the central charge now depends on P_c and the Born-Infeld parameter. The dependence on the Born-Infeld parameter is listed in Table(3.1) which is realised for parameters $Q = 1$, $k = 16\pi$, and $P_c = 15$. The value of the critical central charge increases with the increase in the Born-Infeld parameter. The upper bound is realised for the case of Reissner-Nordstrom AdS black hole, which for the above-mentioned parameters is 36. The next section is devoted to the free energy analysis of the phase transition.

3.2.6 Phase Structure

Here, we shall focus on the phase structure of the considered black hole. The free energy can be computed either from thermodynamic variables using eq.(s)(3.5,3.6) and Bekenstein-Hawking entropy, or from the path integral method. In this regard, we utilise the first method since we already know the thermodynamic variables. The expression of the free energy is given by

$$\begin{aligned} F &= M - TS \\ &= \frac{r_+}{4G} - \frac{r_+^3}{4G\,l^2}\frac{b^2 r_+^3}{6G}\left(1 - \sqrt{1 + \frac{GQ^2}{b^2 r_+^4}}\right) + \frac{2Q^2}{3r_+}\,{}_2F_1\left[\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, -\frac{GQ^2}{b^2 r_+^4}\right] . \end{aligned} \tag{3.53}$$

In extended black hole thermodynamics, the free energy of a charged black hole is determined by its temperature (T), pressure (P), and charge (Q). However, in mixed black hole thermodynamics, it also depends on the central charge (C). Considering the presence of the Born-Infeld parameter in the first law for Born-Infeld black holes, the general free energy can be denoted as $F \equiv F(T, P, Q, C, b)$. To illustrate its behavior concerning the central charge, we present the plot for three distinct values in Fig.(3.1). The primary observation indicates a swallowtail pattern when the central charge exceeds the critical value ($C_c = 35.4769$), while it disappears below this threshold. This behavior is indicative of van der Waals fluid-like characteristics, where, beyond the critical point (in the central charge parameter space, with the remaining parameters fixed), a first-order phase transition from a small to a large black hole occurs.

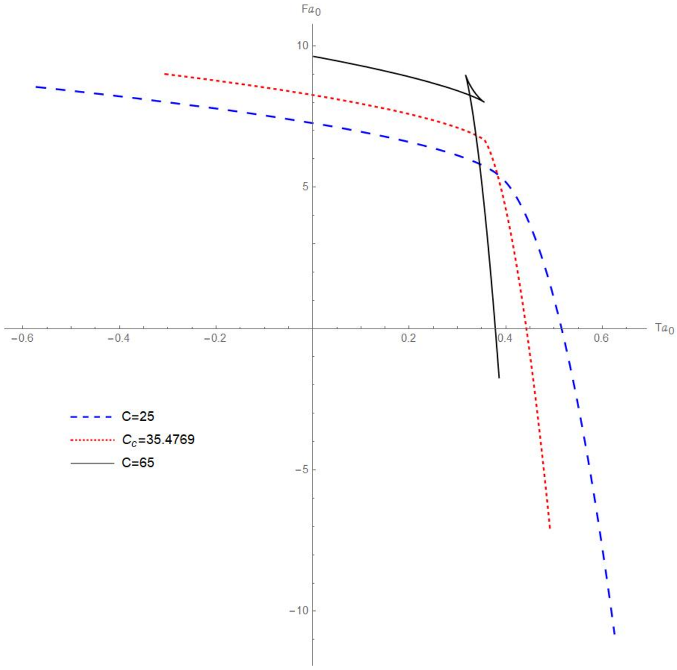


Figure 3.1: Free energy vs Temperature: *Parameters* Q=1; k=16π; P=15; b=30 *(a_0 is a scaling parameter)*: a) $C = 25$ (*Dashed line*), b) $C = C_c = 35.4769$ (*Dotted line*) , c) $C = 60$ (*Solid line*)

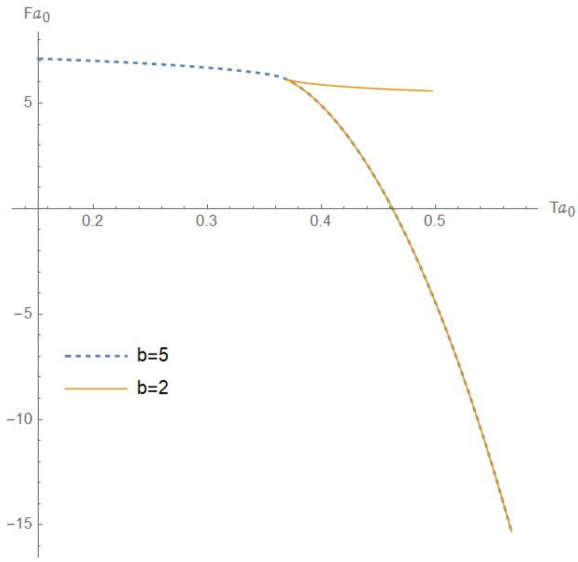


Figure 3.2: Free energy vs Temperature: *Parameters* Q=1; k=16π; P=15; C=30 *(a_0 is a scaling parameter)*: a) $b = 5$ (*Dashed line*), b) $b = 2$ (*Solid line*)

The primary objective of this analysis is to comprehend the influence of the Born-Infeld parameter on this phase transition. In Fig. (s)(3.2,3.3), we have plotted the free energy against the Hawking temperature, both below and above the critical point, for various values of the parameter b. Notably, a common observation in both figures is the behavior at sufficiently low values of the parameter b.

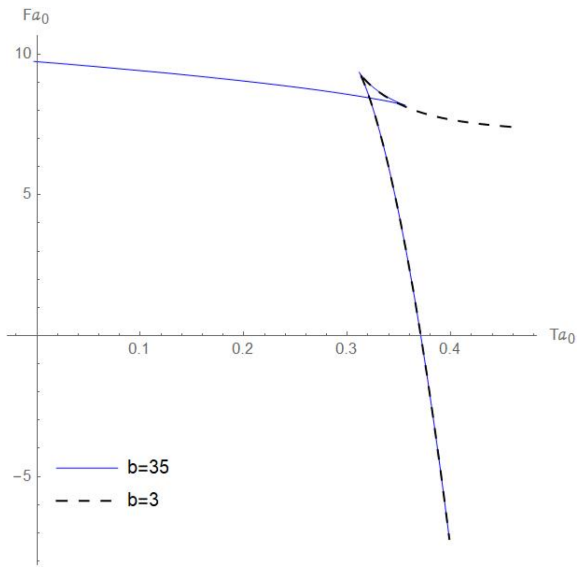


Figure 3.3: Free energy vs Temperature: *Parameters* Q=1; k=16π; P=15; C=70 *(*a_0 *is a scaling parameter)*: a) $b = 30$ (*Dashed line*), b) $b = 3$ (*Solid line*)

For values of b below a certain threshold, the graph does not terminate at the y-axis, exhibiting the emergence of a cusp. The presence of this cusp indicates the absence of black holes (in a thermodynamic sense) within a small temperature range. The two branches associated with the cusp represent unstable and stable black hole solutions, determined by the relative free energy values. The minimum temperature values in both cases are approximately $Ta_0 \sim 0.36$ $(b = 2)$ in Fig. (3.2) and $Ta_0 \sim 0.32$ $(b = 3)$ in Fig. (3.3).

Consequently, below a certain threshold of the parameter b, the system lacks a van der Waals fluid-like phase transition. Therefore, the parameter significantly alters the phase structure of the black holes. In the subsequent section, we will examine black hole solutions in Gauss-Bonnet gravity, applying the same perspective of mixed thermodynamics.

3.3 Review of the Thermodynamics of Gauss-Bonnet *AdS* Black Holes

Now we turn to the black holes in the Gauss-Bonnet gravity. In this section, a brief review of charged AdS black holes in Gauss-Bonnet gravity, along with its thermodynamic properties, is provided. We shall begin with the following action in general D-dimensional *AdS* spacetime

[61]

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\left[R - 2\Lambda + \alpha L_{GB} + L(F)\right] \quad . \tag{3.54}$$

Here, the Gauss-Bonnet Lagrangian density (L_{GB}) takes the following form

$$L_{GB} = R^2 - 4R_{\gamma\delta}R^{\gamma\delta} + R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma}. \tag{3.55}$$

The term, $L(F) = -F^{\mu\nu}F_{\mu\nu}$ represents the Maxwell field Lagrangian density, where $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and Λ denotes the cosmological constant. The parameter, α alongside L_{GB} denotes the Gauss-Bonnet parameter. A static, spherically symmetric black hole solution of the action in eq.(3.54) is as follows [61]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2} \,, \tag{3.56}$$

where

$$f(r) = 1 + \frac{r^2}{2\alpha'}\left(1 - \sqrt{1 - \frac{4\alpha'}{l^2} + \frac{4\alpha' m}{r^{D-1}} - \frac{4\alpha' q^2}{r^{2D-4}}}\right) \quad . \tag{3.57}$$

The parameter m in the solution is related with the AdM mass (M) in the following manner

$$M = \frac{(D-2)\omega_{D-2}}{16\pi G}m \;\; ; \;\; \omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma(D-1)/2} \tag{3.58}$$

where ω_{D-2} is the unit sphere volume in $D-2$ dimensions. Other parameters, denoted as α', and q in eq.(3.57) are related to the parameters α and charge Q of the black hole as

$$\alpha' = (D-3)(D-4)\alpha, \qquad Q = \sqrt{\frac{2(D-2)(D-3)}{G}}\frac{\omega_{D-2}q}{8\pi} \;. \tag{3.59}$$

Mass of the black hole in terms of event horizon radius (r_+) can be obtained from the relation $f(r_+) = 0$. The expression obtained as such is given by

$$M = \frac{(D-2)\omega_{D-2}}{16\pi G}\left(\frac{r_+^{D-1}}{l^2} + r_+^{D-3} + \alpha' r_+^{D-5} + \frac{q^2}{r_+^{D-3}}\right) \;. \tag{3.60}$$

Also, the Hawking temperature can be obtained from the lapse function in the following manner

$$
\begin{aligned}
T &= \frac{1}{4\pi}\frac{\partial f}{\partial r}\bigg|_{r=r_+} \\
&= \frac{(D-1)r_+^3}{4\pi l^2(r_+^2+2\alpha')} + \frac{(D-3)r_+}{4\pi(r_+^2+2\alpha')} + \frac{(D-5)\alpha'}{4\pi r_+(r_+^2+2\alpha')} - \frac{(D-3)q^2 r_+^{7-2D}}{4\pi(r_+^2+2\alpha')}\,.
\end{aligned}
\tag{3.61}
$$

The Gauss-Bonnet term has no effect on the equation of motion in 4 spacetime dimensions and the case becomes non-trivial only for $D \geq 5$. In this section, we have considered the case of $D = 5$ for detailed analysis of phase structure. The Hawking temperature for $D = 5$, takes the form

$$
T_5 = \frac{r_+^3}{\pi l^2(r_+^2+2\alpha')} + \frac{r_+}{2\pi(r_+^2+2\alpha')} - \frac{q^2 r_+^{-3}}{2\pi(r_+^2+2\alpha')}\,. \tag{3.62}
$$

The black hole entropy formula in case of Gauss-Bonnet black hole is not simply the Bekenstein-Hawking formula. In this case, the black hole entropy can be calculated by using the first law of black hole thermodynamics ($dM = TdS$). The entropy fromula becomes

$$
S = \int_0^{r_+} T^{-1}\left(\frac{\partial M}{\partial r_+}\right) dr_+ = \frac{\omega_{D-2}}{4G} r_+^{D-2}\left[1 + \frac{(D-2)2\alpha'}{(D-4)r_+^2}\right]. \tag{3.63}
$$

This completes the list of all the thermodynamic variables associated to the black hole. The aim is to vary the Newton's constant and write the mixed first law using gauge/gravity dictionary as done in case of Born-Infeld black holes. Next section is devoted to this task.

3.3.1 Modified First Law for Gauss-Bonnet AdS Black Holes

We shall follow the same scheme of integrating the variation of Newton's constant in first law, as done for the case of Born-Infeld black holes in the last section. The key changes will be the inclusion of the Gauss-Bonnet parameter in the first law. It should be included because the parameter is dimensionful. We start with redefining the area term in entropy as

$$
A = \omega_{D-2} r_+^{D-2} + \frac{2(D-2)\alpha'}{D-4}\omega_{D-2} r_+^{D-4} \tag{3.64}
$$

such that, entropy can be expressed as $S = \dfrac{A}{4G}$. This leads us to consider mass (M) of the most general black hole as a function of parameter (A), angular meomentum (J), charge (Q), Newton's constant (G), cosmological constant (Λ) and the Gauss-Bonnet parameter (α). That is, $M \equiv M(A, J, Q, \Lambda, G, \alpha)$. Variation of mass with these variables is given by

$$\delta M = \frac{\partial M}{\partial A}\delta A + \frac{\partial M}{\partial G}\delta G + \frac{\partial M}{\partial J}\delta J + \frac{\partial M}{\partial Q}\delta Q + \frac{\partial M}{\partial \Lambda}\delta \Lambda + \frac{\partial M}{\partial \alpha}\delta \alpha. \tag{3.65}$$

The conjugate variables can be read from eq.(3.9). These variables are given by

$$\frac{\partial M}{\partial A} = \frac{\kappa}{8\pi G}\ , \ \ \frac{\partial M}{\partial J} = \Omega\ , \ \ \frac{\partial M}{\partial Q} = \Phi\ , \ \ \frac{\partial M}{\partial \Lambda} = -\frac{V}{8\pi G}\ . \tag{3.66}$$

We define the remaining two, that is, conjugate to G and α in the following manner

$$\frac{\partial M}{\partial G} \equiv -\frac{\zeta}{G}\ , \ \ \frac{\partial M}{\partial \alpha} \equiv \mathcal{A}\ . \tag{3.67}$$

Thus, the eq.(3.65) becomes

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q - \frac{V}{8\pi G}\delta\Lambda - \zeta\frac{\delta G}{G} + \mathcal{A}\delta\alpha\ . \tag{3.68}$$

Next, we wish to calculate the variable conjugate to G in terms of the rest of the black hole parameters. We start with the modified mass which is a function of the parameter α as well. The modified mass form is

$$GM = \mathcal{M} = \mathcal{M}(A, GJ, \sqrt{G}Q, \Lambda, \alpha)\ . \tag{3.69}$$

Differential on both sides of the above equation leads us to

$$\begin{aligned} G\delta M + M\delta G = &\frac{\partial \mathcal{M}}{\partial A}\delta A + J\frac{\partial \mathcal{M}}{\partial (GJ)}\delta G + G\frac{\partial \mathcal{M}}{\partial (GJ)}\delta J + \\ &\sqrt{G}\frac{\partial \mathcal{M}}{\partial (\sqrt{G}Q)}\delta Q + \frac{Q}{2\sqrt{G}}\frac{\partial \mathcal{M}}{\partial (\sqrt{G}Q)}\delta G + \frac{\partial \mathcal{M}}{\partial \Lambda}\delta\Lambda + \frac{\partial \mathcal{M}}{\partial \alpha}\delta\alpha\ . \end{aligned} \tag{3.70}$$

Minor rearrangements in the above equation, in order to facilitate comparison with the mass variation eq.(3.68) gives us

$$\begin{aligned}\delta M = &\frac{1}{G}\frac{\partial\mathcal{M}}{\partial A}\delta A + \frac{\partial\mathcal{M}}{\partial(GJ)}\delta J + \frac{1}{\sqrt{G}}\frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)}\delta Q + \frac{\partial\mathcal{M}}{\partial\Lambda}\frac{\delta\Lambda}{G} \\ &+ \frac{\partial\mathcal{M}}{\partial\alpha}\frac{\delta\alpha}{G} + \left[-\frac{M}{G} + \frac{Q}{2G^{\frac{3}{2}}}\frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)} + \frac{J}{G}\frac{\partial\mathcal{M}}{\partial(GJ)}\right]\delta G\ .\end{aligned} \tag{3.71}$$

The form of conjugate variables after comparison is

$$\begin{aligned}&\frac{\partial\mathcal{M}}{\partial A} = \frac{\kappa}{8\pi}\ ,\ \frac{\partial\mathcal{M}}{\partial(GJ)} = \Omega\ ,\ \frac{\partial\mathcal{M}}{\partial(\sqrt{G}Q)} = \sqrt{G}\Phi\ , \\ &\frac{\partial\mathcal{M}}{\partial\Lambda} = -\frac{V}{8\pi}\ ,\ \frac{1}{G}\frac{\partial\mathcal{M}}{\partial\alpha} = \mathcal{A}\ .\end{aligned} \tag{3.72}$$

Thus, we obtain the analytical form of the parameter ζ in terms of other black hole parameters. The form of ζ is

$$\zeta = M - \frac{Q\Phi}{2} - \Omega J\ . \tag{3.73}$$

It may appear that ζ is independent of parameter α, but that is not the case. ζ depends on α as mass M depends on α. We ultimately want to write the mixed form of the first law, for which we replace the G variation in eq.(3.68) by using eq.(3.21). The substitution leads us to

$$\begin{aligned}\delta M =&\frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q - \frac{V}{8\pi G}\delta\Lambda + \mathcal{A}\delta\alpha + \frac{2\zeta}{DC}\delta C \\ &+\frac{D-2}{D}\zeta\frac{\delta P}{P} \\ =&T\delta S + \Omega\delta J + \Phi\delta Q + \left[\frac{2\zeta}{DC} - \frac{2(TS+PV)}{DC}\right]\delta C \\ &+\mathcal{A}\delta\alpha + \left[V + \frac{D-2}{DP}\zeta - \frac{D-2}{DP}(TS+PV)\right]\delta P\ ,\end{aligned} \tag{3.74}$$

which can also be rewritten as

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q + \mathcal{A}\delta\alpha + V_{\mathcal{C}}\delta P + \mu_{\mathcal{C}}\delta C\ . \tag{3.75}$$

Here, $\mu_{\mathcal{C}}$ and $V_{\mathcal{C}}$ are the effective thermodynamic chemical potential and thermodynamic volume. The form of these variables is given by

$$V_{\mathcal{C}} = V + \frac{D-2}{DP}\zeta - \frac{D-2}{DP}(TS+PV)\,, \tag{3.76}$$

$$\mu_{\mathcal{C}} = \frac{2\zeta}{D\mathcal{C}} - \frac{2(TS+PV)}{D\mathcal{C}}\,. \tag{3.77}$$

Eq.(3.75) is the required mixed first law. V_c is the volume conjugate to Λ variation now (pressure, to be exact). The physical interpretation is still a matter of more research, however, this should be the form of volume once holographic system is studied using Gauss-Bonnet black holes. We will show the explicit dependence of parameter α on the effective volume in the next section.

3.3.2 Smarr Relation in Extended Thermodynamics

Following the strategy similar to the case of Born-Infeld black holes, we derive the Smarr relation here for Gauss-Bonnet black hole using a scaling argument. The aim is to express parameter V in terms of other black hole variables so that it can be used to find the expression of V_c. We begin with the first law in the extended phase space, which takes the following form

$$\delta M = T\delta S + \Phi\delta Q + \Omega\delta J + V\delta P + \mathcal{A}\delta\alpha\,. \tag{3.78}$$

Mass of the black hole is considered as a function of $M \equiv M(S,J,P,Q,\alpha)$. In general D spacetime dimensions, these quantities have the following dimensions in terms of length (L)

$$\begin{aligned}
&[M] = L^{D-3},\ [S] = L^{D-2},\ [J] = L^{D-2}\,,\\
&[P] = L^{-2}\,,\ [Q] = L^{D-3},\ [\alpha] = L^{2}\,.
\end{aligned} \tag{3.79}$$

Using Euler's theorem of quasi-homogeneous functions lead us to the following relation

$$\begin{aligned}
(D-3)M = (D-2)S\frac{\delta M}{\delta S} + 2\alpha\frac{\delta M}{\delta\alpha} - 2P\frac{\delta M}{\delta P}\\
+ (D-3)Q\frac{\delta M}{\delta Q} + (D-2)J\frac{\delta M}{\delta J}\,.
\end{aligned} \tag{3.80}$$

From first law in eq.(3.78), it is clear that

$$T = \frac{\delta M}{\delta S}, \quad \Phi = \frac{\delta M}{\delta Q}, \quad V = \frac{\delta M}{\delta P}, \quad \Omega = \frac{\delta M}{\delta J}, \quad \mathcal{A} = \frac{\delta M}{\delta \alpha}. \tag{3.81}$$

On substitution of these conjugate variables in eq.(3.28), we obtain the required Smarr relation as

$$(D-3)M = (D-2)TS + 2\mathcal{A}\alpha - 2PV + (D-3)\Phi Q + (D-2)\Omega J\ . \tag{3.82}$$

The explicit expression of V can be obtained by rearranging the above relation. The form of the expression is as follows

$$V = \frac{D-2}{2P}TS + \frac{\mathcal{A}\alpha}{P} + \frac{D-3}{2P}\Phi Q - \frac{D-3}{2P}M + \frac{D-2}{2P}\Omega J. \tag{3.83}$$

Now, we are in position to write the expression of new volume V_c in terms of other thermodynamic parameters. The expression takes the form

$$V_C = \frac{2M + 4\Gamma\alpha + (D-4)Q\Phi}{2DP}\ . \tag{3.84}$$

As mentioned earlier, the new thermodynamic volume V_c does dependent on the Gauss-Bonnet parameter α; hence, it differs form that of the expression of volume in Einstein's gravity. Similar to the approach used in the last section for the case of Born-Infeld black holes, we will now concentrate on the phase transition properties of the black hole. The emphasis will be on characterising van der Waals fluid-like behaviour through free energy analysis within the new setup of the mixed first law.

3.3.3 Phase Structure

The phase structure under Gauss-Bonnet modifications of the Einstein-Hilbert action is the topic of this section. The focus will be on the dependence of parameter α on the phase transition properties. Using the thermodynamic quantities of the black hole calculated above, we can compute the free energy, which takes the following form

$$
\begin{aligned}
F &= M - TS \\
&= \frac{(D-2)\omega_{D-2}}{16\pi G}\left(\frac{r_+^{D-1}}{l^2} + r_+^{D-3} + \alpha' r_+^{D-5} + \frac{q^2}{r_+^{D-3}}\right) \\
&- \frac{\omega_{D-2} r^{D-3}}{16\pi G(r_+^2 + 2\alpha')}\left(\frac{(D-4)r_+^2 + 2(D-2)\alpha'}{(D-4)r_+^2}\right) \\
&\left(\frac{(D-1)r_+^4}{l^2} + (D-3)r_+^2 + (D-5)\alpha' - \frac{(D-3)q^2}{r_+^{2D-8}}\right) \;.
\end{aligned} \tag{3.85}
$$

We shall study the specific case of five spacetime dimensions. The expression for $D = 5$ becomes

$$
\begin{aligned}
F_5 = \frac{3\pi}{8G}\left(\frac{r_+^4}{l^2} + r_+^2 + \alpha' + \frac{q^2}{r_+^2}\right) - \frac{\pi(r_+^2 + 6\alpha')}{4G(r_+^2 + 2\alpha')} \\
\left(\frac{2r_+^4}{l^2} + r_+^2 - \frac{q^2}{r_+^2}\right)
\end{aligned} \tag{3.86}
$$

The above expression for the free energy is, in general, a function of temperature T, pressure P, central charge C, charge Q, and the Gauss-Bonnet parameter α. Consequently, the free energy, in the context of extended thermodynamics, now depends on the central charge as well. Fig.(3.4) illustrates the free energy plotted against temperature for fixed value of other parameters. The plot clearly exhibits the dependence on the parameter C, revealing a critical value of central charge above which the swallow-tail behaviour persists, indicating a first-order phase transition. Hence, even in Gauss-Bonnet gravity, the qualitative behaviour of phase transition remains the same. The critical point's dependence on the Gauss-Bonnet parameter is a crucial consideration. Selecting a point above the critical value of central charge with the same parameter values, we plotted for different values of α in Fig.(3.5). The plot demonstrates that increasing the value of α shifts the phase transition point to higher critical parameter values. For instance, for $\alpha = 0.01 l_0^2$, the phase transition does not occur at $C = 500 l_0^3$. In the subsequent analysis, we studied the dependence of the phase transition point for various pressure values. Fig.(3.6), shows the free energy plotted with respect to Hawking temperature for several values of pressure while keeping other parameters fixed. The increase in pressure shifts the critical temperature to a higher value, thus demonstrating compliance with the characteristics of a van der Waals fluid.

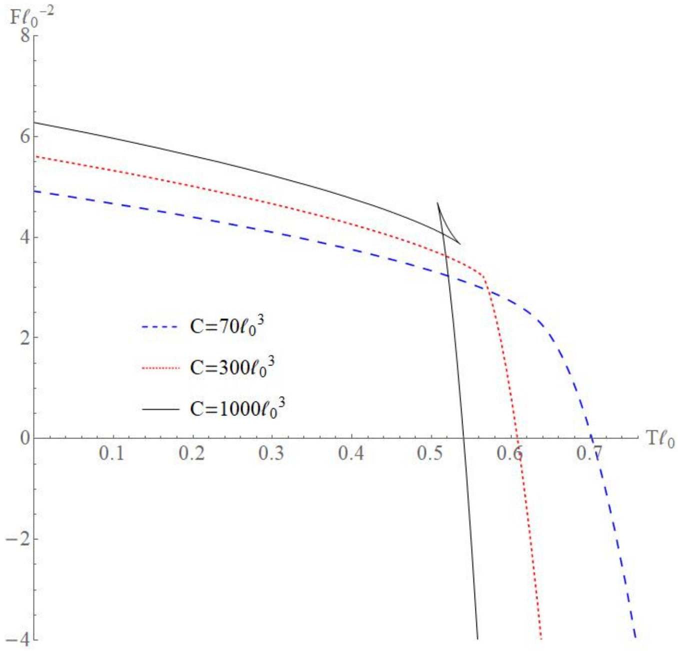


Figure 3.4: Free energy versus temperature for different values of the central charge: $Q = 1.0l_0^2$; $k = 16\pi$; $Pl_0^2 = 15$; $\alpha = 0.001l_0^2$; $D = 5$

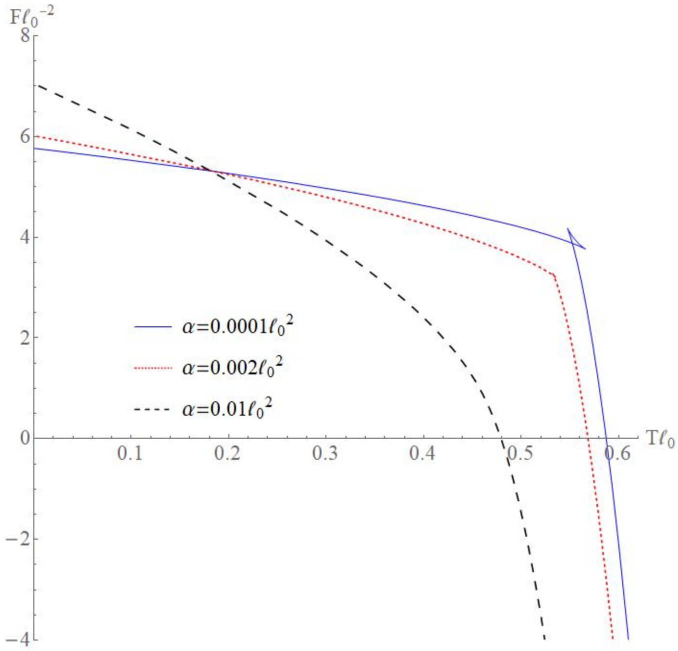


Figure 3.5: Free energy versus temperature for different values of the Gauss-Bonnet parameter: $Q = 1.0l_0^2$; $k = 16\pi$; $Pl_0^2 = 15$; $C = 500l_0^3$; $D = 5$

We have discussed the dependence of the central charge, pressure, and the Gauss-Bonnet parameter on the phase transition point. Considering the analysis conducted in D space-time dimensions, we can also examine the dependence of dimensions on the phase transition. Fig.(s)(3.7,3.8,3.9) show the denpendence of the free energy on temperature for $D = 5,\ 6,\ 7$.

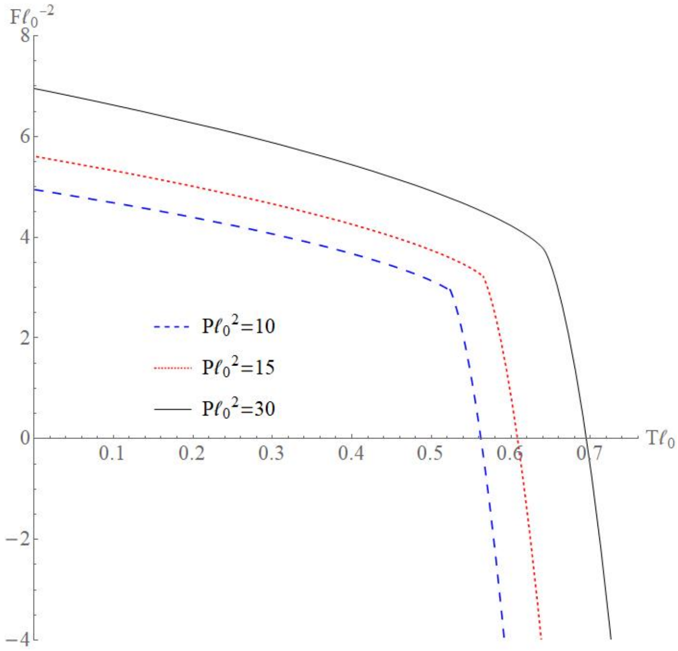


Figure 3.6: Free energy versus temperature for different values of pressure: $Q = 1.0l_0^2$; $k = 16\pi$; $\alpha = 0.001l_0^2$; $C = 300l_0^3$; $D = 5$

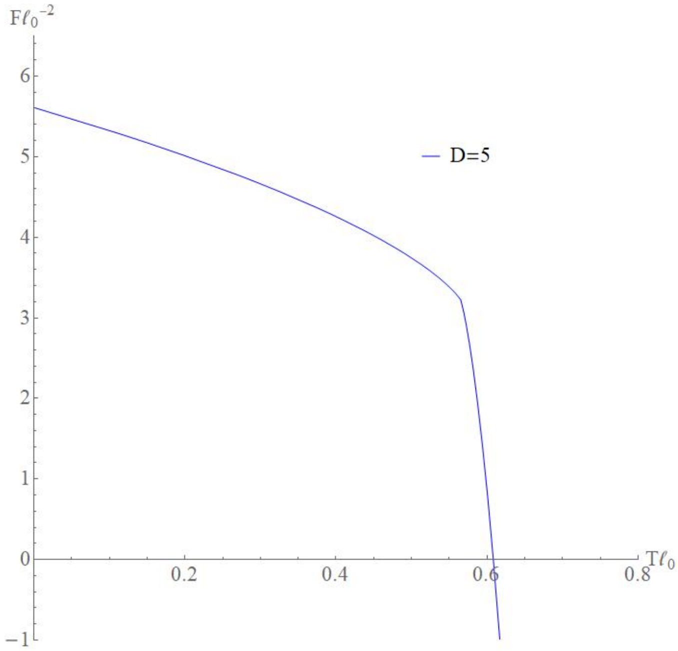


Figure 3.7: Free energy versus temperature in five spacetime dimensions: $Q = 1.0l_0^2$; $k = 16\pi$; $\alpha = 0.001l_0^2$; $C = 300l_0^3$; $D = 5$

Although there is a shift in the phase transition point to higher temperature values, there is no qualitative change. In the next section, we shall calculate the exact critical values of the parameters.

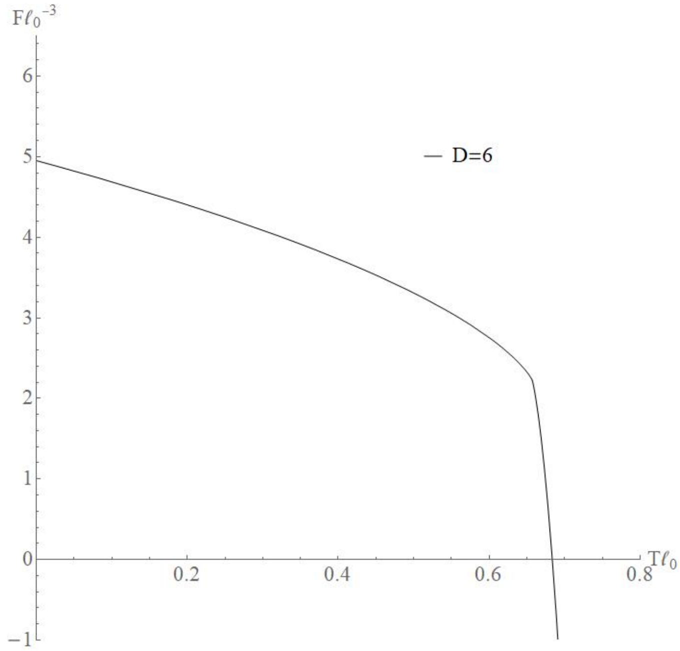


Figure 3.8: Free energy versus temperature in six spacetime dimensions: $Q = 1.0l_0^3$; $k = 16\pi$; $\alpha = 0.001l_0^2$; $C = 450l_0^4$; $D = 6$

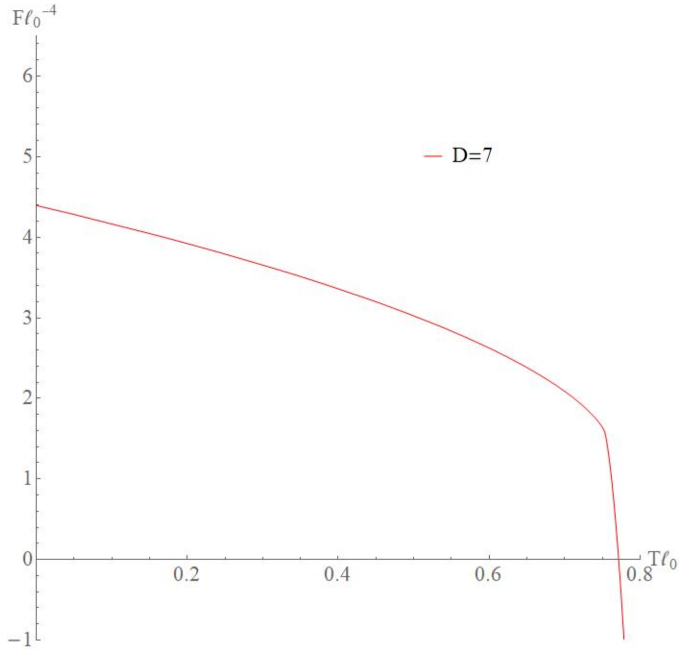


Figure 3.9: Free energy versus temperature in seven spacetime dimensions: $Q = 1.0l_0^4$; $k = 16\pi$; $\alpha = 0.001l_0^2$; $C = 600l_0^5$; $D = 7$

3.3.4 Universality Breaking for Central Charge

So far, we have qualitatively discussed the phase transition behaviour through free energy analysis. Next, we will calculate the critical value of the thermodynamic quantities for $D = 5$

spacetime dimensions. We begin with the following two equations [146]

$$\frac{\partial T_5}{\partial r_+} = 0 \, , \tag{3.87}$$

$$\frac{\partial^2 T_5}{\partial r_+^2} = 0 \, . \tag{3.88}$$

Differentiating eq.(3.62), we get

$$2r_{+(c)}^8 + (12\alpha' - l_c^2)r_{+(c)}^6 + 2l_c^2\alpha' r_{+(c)}^4 + 5l_c^2 q^2 r_{+(c)}^2 + 6l_c^2 q^2 \alpha' = 0 \tag{3.89}$$

and

$$\begin{aligned}(l_c^2 - 4\alpha')r_{+(c)}^8 + (24\alpha'^2 - 6\alpha' l_c^2)r_{+(c)}^6 - 15q^2 l_c^2 r_{+(c)}^4 \\ -34\alpha' q^2 l_c^2 r_{+(c)}^2 - 24\alpha'^2 q^2 l_c^2 = 0 \, .\end{aligned} \tag{3.90}$$

The subscript c with all the quantities denote the values at the critical point. These equations are not easy to solve exactly, and we shall adopt perturbative methods to solve them. We wish to understand the universal characteristics of the central charge and how it depends on the parameter α. We start with a solution of the form

$$r_{+(c)} \cong r_{+(c)}^{(0)} + \alpha' r_{+(c)}^{(1)} \, , \;\; l_c \cong l_c^{(0)} + \alpha' l_c^{(1)} \tag{3.91}$$

which is first order in α. Keeping these back in eq.(s)(3.89, 3.90), we get the forms of $r_{+(c)}^{(0)}$ and $l_c^{(0)}$ to be

$$r_{+(c)}^{(0)} = 15^{1/4}\sqrt{q} \, , \;\; l_c^{(0)} = 3^{3/4}5^{1/4}\sqrt{q} \, . \tag{3.92}$$

Next, we calculate the values at the first order. Starting with the values of $r_{+(c)}^{(0)}$ and $l_c^{(0)}$ and plugging these back in eq.(s)(3.87,3.88), we obtain the following forms of $r_{+(c)}^{(1)}$ and $l_c^{(1)}$

$$r_{+(c)}^{(1)} = \frac{3^{\frac{3}{4}}4}{5^{\frac{5}{4}}\sqrt{q}} \, , \;\; l_c^{(1)} = \frac{3^{\frac{1}{4}}24}{5^{\frac{5}{4}}\sqrt{q}} \, . \tag{3.93}$$

The parameter q and the net electric charge Q on the black hole in five dimensional AdS spacetime are connected as

$$q = 2\frac{\sqrt{G}Q}{\sqrt{3}\pi} \ . \tag{3.94}$$

Using the above form of q in $D = 5$, the critical values of r_+ and l to first order in α become

$$r_{+(c)} = 5^{\frac{1}{4}}\sqrt{\frac{2}{\pi}}\sqrt{Q}G^{\frac{1}{4}} + \frac{12\alpha\sqrt{2\pi}}{5^{\frac{5}{4}}\sqrt{Q}G^{\frac{1}{4}}} \ , \tag{3.95}$$

$$l_c = 5^{\frac{1}{4}}\sqrt{\frac{6}{\pi}}\sqrt{Q}G^{\frac{1}{4}} + \frac{24\sqrt{6\pi}\alpha}{5^{\frac{5}{4}}\sqrt{Q}G^{\frac{1}{4}}} \ . \tag{3.96}$$

These values can be used to calculate the thermodynamic quantities like pressure, temperature and volume V_c. However, we are more interested in knowing the critical value of the central charge C. Substituting value of l_c in eq.(3.10), the critical central charge value up to first order in α becomes

$$\begin{aligned} C_c &= k\frac{l_c^3}{16\pi G} \\ \implies C_c &\cong k\frac{l_c^{(0)^3} + 3\alpha' l_c^{(1)}}{16\pi G} = k\frac{3^{\frac{3}{2}}5^{\frac{3}{4}}Q^{\frac{3}{2}}}{(2\pi)^{\frac{5}{2}}G^{\frac{1}{4}}} + \alpha k\frac{27\sqrt{6}\sqrt{Q}}{5^{\frac{3}{4}}G^{\frac{3}{4}}\pi^{\frac{3}{2}}} \ . \end{aligned} \tag{3.97}$$

This is the most important finding so far. The critical value not only depends on the charge Q of the black hole but depends also on Newton's constant G to zeroth as well as first order in α. Thus, the universality [146] breaks for this case as well.

Next, we address a more general question. Does the universal critical value have any implications for the dimensions and modified gravity theories considered? In other worlds, is it a generic feature of Einstein-Maxwell theory in four dimensions? As seen in the last two cases, any modification leads to breaking of the universal character. The following section presents the criticality analysis for higher dimensional Einstein's theory.

3.4 Universal Nature of Critical Central Charge in Higher Dimensions

We shall start with the Hawking temperature associated to a black hole geometry in general D-dimensional Einstein gravity. It can be obtained from eq.(3.61) by taking the limit $\alpha \to 0$.

The temperature expression becomes

$$T = \frac{1}{4\pi}\left(\frac{(D-1)r_+}{l^2} + \frac{(D-3)}{r_+} - \frac{(D-3)q^2}{r_+^{2D-5}}\right) . \tag{3.98}$$

Following the scheme used in the last two sections, we locate the critical point of the phase transition using following set of equations

$$\frac{\partial T}{\partial r_+} = \frac{\partial^2 T}{\partial r_+^2} = 0 . \tag{3.99}$$

These equations are easy to solve exactly. The solution is given by

$$r_c = \left[(2D-5)(D-2)q^2\right]^{1/(2D-6)} , \text{ and } l_c = h(D)q^{1/(D-3)} , \tag{3.100}$$

where

$$h(D) = \frac{(D-1)^{1/2}(2D-5)^{1/2(D-3)}(D-2)^{(D-2)/2(D-3)}}{(D-3)} . \tag{3.101}$$

Using these values, we can calculate the critical central charge from eq.(3.10), which takes the form

$$C_c = \kappa\frac{g(D)G^{(D-2)/2(D-3)}Q^{(D-2)/(D-3)}}{16\pi G} . \tag{3.102}$$

Here, the dimensional dependent factor $g(D)$ is

$$g(D) = h^{D-2}\left(\frac{8\pi}{\omega_{D-2}\sqrt{2(D-2)(D-3)}}\right)^{\frac{(D-2)}{(D-3)}} . \tag{3.103}$$

The eq.(3.102) is independent of G only for $D = 4$. This answers the question of universal nature of critical central charge being a special feature of $4-D$ Einstein-Maxwell theory.

3.5 Conclusion

This chapter has been dedicated to examining the thermodynamics of black holes in the extended phase space, where Newton's constant is treated as a thermodynamic parameter along with the cosmological constant. Through the use of the holographic dictionary, we incorporated the boundary variable, the central charge of the theory (C), into the first law, referred to as the mixed first law. We delved into the investigation of phase transition properties in various scenarios, including Born-Infeld black holes in four spacetime dimensions, Gauss-Bonnet black holes in general D dimensions ($D \geq 5$), and Reissner-Nordstrom black holes in general D dimensions. While earlier studies had revealed a distinctive universality in the critical value of the central charge concerning Reissner-Nordstrom black holes in four spacetime dimensions [146], our analysis highlights that this universality is exclusive to the context of Einstein-Maxwell theory in four spacetime dimensions and does not extend to the cases considered.

We derived the mixed first law for Born-Infeld and Gauss-Bonnet black holes, noting that the variable conjugate to the thermodynamic pressure undergoes modifications due to the Born-Infeld and Gauss-Bonnet parameters. Our comprehensive study thoroughly explored the impact of these parameters on the phase transition point. Notably, we observed qualitative behavior akin to that of van der Waals fluids across all cases, with quantitative changes, including shifts in the phase transition point.

CHAPTER 4

Summary and Future Work

4.1 Thesis Summary

This thesis focused on an in-depth exploration of the thermodynamic stability and phase transition properties of black holes. Specifically, we investigated the thermodynamics of Einstein's black holes in Born-Infeld electromagnetism and black holes in Gauss-Bonnet gravity, aiming to grasp the implications resulting from the introduction of these modifications.

The first section delved into the thermodynamics of these black holes in constant AdS spacetime in general D-spacetime dimensions. After a concise review of their thermodynamic properties, we analyzed the behaviour of the heat capacity, revealing a finite number of infinite discontinuities. Notably, the heat capacity exhibited both negative and positive values, dependent on the range of black hole masses, with an infinite discontinuity observed between a stable and an unstable phase, similar to the behavior of black holes in General Relativity in flat spacetime. Furthermore, we addressed the challenge of determining the order of the phase transition. Through the application of two well-known methods from standard thermodynamics - the Ehrenfest scheme and Ruppeiner state space geometry analysis - we established the second-order nature of the phase transition for both the Born-Infeld and Gauss-Bonnet black holes.

In contrast to black holes in Einstein's theory with a cosmological constant, we observed no qualitative change in the phase transition nature, although the non-linearity parameter and the Gauss-Bonnet parameter did induce quantitative shifts in the phase transition points. Thus, our study concluded that the thermodynamic nature of black holes remains unaffected by higher curvature terms in the action and electromagnetic non-linearity.

Chapter 3 of the thesis was concerned with a relativity new understanding in black hole thermodynamics, called the Extended Black Hole Thermodynamics. We considered the study of phase structure of black holes in extended thermodynamics with incorporation of the gauge/gravity duality principle. Our analysis of Born-Infeld and Gauss-Bonnet black holes within this framework, with Newton's constant as a thermodynamic variable, demonstrated the persistence of van der Waals fluid-like behavior and a small-to-large black hole phase transition with increasing temperature. Notably, we discovered the breakdown of the critical central charge universality in both cases. We also proved an important result that the universality of critical central charge is a generic feature of the Einstein-Maxwell theory in four spacetime dimensions only.

4.2 Future Work

The problem of thermodynamic stability of a black hole is still relevant, regardless of the asymptotic structure of the spacetime. A needed microscopic theory is not only to explain the Bekenstein-Hawking entropy formula but also the phase structure discussed in this thesis. The existence of infinite discontinuity and unstable phases make black holes significantly distinct form any standard thermodynamic system. In order to address these issues, following are the directions we aim to explore in the future:

- *Isolated Horizon and Phase Structure:* A stationary black hole solution requires the existence of a time-like killing vector throughout the spacetime, and this condition is essential in defining thermodynamic equilibrium. However, the condition is highly restrictive and not suitable for comparison to an ordinary thermodynamic system, where we have a system in contact with the rest of the world. A quasi-local approach has been proposed in the literature where the killing vector field is restricted to the near horizon only. This equilibrium condition corresponds to an isolated horizon, which represents a more realist situation. We plan to understand the phase structure and phase transition properties of the isolated horizon. We also aim to extend this study to a dynamical horizon.
- *Euclidean Quantum Gravity:* Gibbons and Hawking formulated the Euclidean action and derived the Bekenstein-Hawking formula from the classical configuration. Various

attempts have also been made to go beyond zeroth-order calculations. The interpretation of partition function obtained from the Euclidean path integral approach in gravity surpasses conventional understanding in field theory. The Hawking-Page phase transition in Schwarzschild black hole and small-large black hole phase transition in charged black holes in a constant AdS background are understood using Euclidean gravity. However, the van der Waals fluid-like behaviour in extended black hole thermodynamics and the small-large black hole phase transition have yet to be comprehended using the Euclidean path integral approach. We aim to make progress in this direction. Euclidean quantum gravity may also have more to teach about thermodynamic stability of black holes.

Bibliography

[1] K. Schwarzschild (translation and foreword by S.Antoci and A.Loinger), On the gravitational field of a mass point according to Einstein's theory, arXiv:physics/9905030 (Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.) 1916 189-196)

[2] Chandrasekhar, S. The Maximum Mass of Ideal White Dwarfs, Provided by the SAO/NASA Astrophysics Data System

[3] J. R. Oppenheimer and H. Snyder, On Continued Gravitational Contraction, Phys. Rev. (2), 56(5):455–459, 1939.

[4] R. P. Geroch, Topology in General Relativity, J. Math. Phys. 8, 782–786 (1967)

[5] R. P. Geroch, Domain of Dependence, J. Math. Phys. 11, 437–449 (1970)

[6] Choquet-Bruhat, Y., Geroch, R., Global aspects of the Cauchy problem in general relativity Commun.Math. Phys. 14, 329–335 (1969)

[7] Kronheimer, E., & Penrose, R. On the structure of causal spaces, Mathematical Proceedings of the Cambridge Philosophical Society, 63(2), 481-501 (1967)

[8] Penrose, R., Republication of: Conformal treatment of infinity Gen Relativ Gravit 43, 901–922 (2011)

[9] A. K. Raychaudhuri, Relativistic Cosmology. I, Phys. Rev. 98, 1123 – 1955

[10] R. Penrose, Gravitational Collapse and Space-Time Singularities, Phys. Rev. Lett. 14, 57 – 1965

[11] E. Witten, Light Rays, Singularities, and All That, Rev. Mod. Phys. 92, 045004 – 2020

[12] S.W. Hawking and R. Penrose, The Singularities of Gravitational Collapse and Cosmology, Proc. Roy. Soc. Lond. A314, 529 (1970).

[13] S. Hawking, G. F. R. Ellis The Large Scale Structure of Space-Time. Cambridge: Cambridge University Press (1973)

[14] R. Penrose, Gravitational Collapse: The Role of General Relativity, Rev. del Nuovo Cim. 1, 252 (1969)

[15] Penrose, R., Floyd, R. Extraction of Rotational Energy from a Black Hole Nature Physical Science 229, 177–179 (1971)

[16] S. W. Hawking, Gravitational Radiation from Colliding Black Holes, Phys. Rev. Lett. 26, 1344 - 1971

[17] J. M. Bardeen, B. Carter and S.W. Hawking, The Four Laws of Black Hole Mechanics, Comm. Math. Phys. 31, 161 (1973).

[18] Bekenstein, J.D., Black holes and the second law, Lett. Nuovo Cimento 4, 737–740 (1972)

[19] Bekenstein, J.D., Black holes and entropy, Phys. Rev. D 7, 2333 – Published 15 April 1973

[20] S. Hawking, Black hole explosions? Nature 248, 30–31 (1974).

[21] S. Hawking, Particle creation by black holes.Commun.Math. Phys. 43, 199–220 (1975)

[22] J. D. Bekenstein, Statistical black-hole thermodynamics, Phys. Rev. D 12, 3077 (1975)

[23] S. W. Hawking, Black holes and thermodynamics, Phys. Rev. D 13, 191 (1976)

[24] T. Regge and J. A. Wheeler, Stability of a Schwarzschild Singularity, Phys. Rev. 108, 1063 – 1957

[25] C. V. Vishveshwara, Stability of the Schwarzschild Metric, Phys. Rev. D 1, 2870 - 1970

[26] A. Strominger, C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy, Physics Letters B, Volume 379, Issues 1–4, 1996

[27] A. Dabholkar, Exact Counting of Supersymmetric Black Hole Microstates, Phys. Rev. Lett. 94, 241301 – 2005

[28] C. Rovelli, Black Hole Entropy from Loop Quantum Gravity, Phys. Rev. Lett. 77, 3288 – 1996

[29] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Quantum Geometry and Black Hole Entropy, Phys. Rev. Lett. 80, 904 – 1998

[30] G. A. Vilkovisky, Effective action in quantum gravity, Class. Quantum Grav. 9 895 1992

[31] I.L. Buchbinder, S. Odintsov, L. Shapiro, Effective Action in Quantum Gravity, Taylor and Francis Group

[32] I. Antoniadis, E. Gava, K.S. Narain, Moduli corrections to gauge and gravitational couplings in four-dimensional superstrings, Nuclear Physics B, Volume 383, Issues 1–2, 1992

[33] J. F. Donoghue, General relativity as an effective field theory: The leading quantum corrections, Phys. Rev. D 50, 3874 – 1994

[34] J. F. Donoghue, The effective field theory treatment of quantum gravity, AIP Conf. Proc. 1483, 73–94 (2012)

[35] T. Clifton, P. G. Ferreira, A. Padilla, C. Skordis, Modified gravity and cosmology, Physics Reports, Volume 513, Issues 1–3, 2012

[36] D. Germanis, General Relativity and Modified Gravity, Imperial College London

[37] Born, M., Infeld, L., Foundations of the New Field Theory, Nature 132, 1004 (1933)

[38] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. 12 (1971) 498–501

[39] R. Adamson, Black holes and thermodynamics, University of Toronto 2003

[40] R. Wald, Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics, Chicago Lectures in Physics 1995

[41] R. Wald, General Relativity, University of Chicago Press 1984

[42] L. Smarr, Mass Formula for Kerr Black Holes, Phys. Rev. Lett. 30, 71 – Published 8 January 1973; Erratum Phys. Rev. Lett. 30, 521 (1973)

[43] P. C. W. Davies, The Thermodynamic Theory of Black Holes. Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, 353(1675), 499–521

[44] P. C. W. Davies, Thermodynamics of black holes, Rep. Prog. Phys. 41 1313 1978

[45] M. E. Peskin, D. V. Schroeder, An Introduction to Quantum Field Theory, Taylor and Francis group 1990

[46] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D 15, 2752 – 1977

[47] S. W. Hawking, Quantum gravity and path integrals, Phys. Rev. D 18, 1747 – Published 15 September 1978

[48] N. Banerjee, S. Banerjee, R. K. Gupta, I. Mandal and A. Sen, Supersymmetry, Localization and Quantum Entropy Function, J. High Energ. Phys. 2010, 91 (2010)

[49] S. Banerjee, R. K. Gupta, and A. Sen, Logarithmic corrections to extremal black hole entropy from quantum entropy function, J. High Energ. Phys. 2011, 147 (2011)

[50] S. Banerjee, R. K. Gupta, I. Mandal, and A. Sen, Logarithmic corrections to $\mathcal{N} = 4$ and $\mathcal{N} = 8$ black hole entropy: a one loop test of quantum gravity,J. High Energ. Phys. 2011, 143 (2011)

[51] J. W. York, Black-hole thermodynamics and the Euclidean Einstein action, Phys. Rev. D 33, 2092 – 1986

[52] J. D. Brown, J. W. York, The Path Integral Formulation of Gravitational Thermodynamics, Based on the talk presented by J.D. Brown at the conference The Black Hole 25 Years After, Santiago, Chile, January 1994

[53] S. W. Hawking, D. N. Page, Thermodynamics of black holes in anti-de Sitter space, Commun.Math. Phys. 87, 577–588 (1983)

[54] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Charged AdS black holes and catastrophic holography, Phys. Rev. D 60, 064018 – 1999

[55] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, Holography, thermodynamics, and fluctuations of charged AdS black holes, Phys. Rev. D 60, 104026 – 1999

[56] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, 061102 – Published 11 February 2016

[57] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. 119, 161101 – 2017

[58] Emparan, R., Reall, H.S., Black Holes in Higher Dimensions Living Rev. Relativ. 11, 6 (2008)

[59] C. Garraffo and G. Giribet, The Lovelock Black holes, Modern Physics Letters A Vol. 23, No. 22, pp. 1801-1818 (2008)

[60] D. G. Boulware and S. Deser, String-Generated Gravity Models, Phys. Rev. Lett. 55, 2656 – 1985

[61] D.L. Wiltshire, Spherically symmetric solutions of Einstein-Maxwell theory with a Gauss-Bonnet term, Physics Letters B, Volume 169, Issue 1, 1986

[62] R. C. Myers and J. Z. Simon, Black-hole thermodynamics in Lovelock gravity Phys. Rev. D 38, 2434 – 1988

[63] Rong-Gen Cai, Gauss-Bonnet black holes in AdS spaces, Phys. Rev. D 65, 084014 – 2002

[64] Y. M. Cho and Ishwaree P. Neupane, Anti–de Sitter black holes, thermal phase transition, and holography in higher curvature gravity, Phys. Rev. D 66, 024044 – 2002

[65] Tim Clunan et al., On Gauss–Bonnet black hole entropy Class. Quantum Grav. 21 3447 2004

[66] Myung, Y.S., Kim, YW. & Park, YJ., Thermodynamics of Gauss–Bonnet black holes revisited, Eur. Phys. J. C 58, 337–346 (2008)

[67] Alexeyev, S., Popov, N., Startseva, M. et al., Kerr-Gauss-Bonnet black holes: Exact analytical solution, J. Exp. Theor. Phys. 106, 709–713 (2008)

[68] I. P. Neupane and N. Dadhich, Entropy bound and causality violation in higher curvature gravity, Class. Quantum Grav. 26 015013 2009

[69] Akbar, M., Generalized Second Law of Thermodynamics in Extended Theories of Gravity, Int J Theor Phys 48, 2672–2678 (2009)

[70] Camanho, X.O., Edelstein, J.D., Maldacena, J. et al., Causality constraints on corrections to the graviton three-point coupling, J. High Energ. Phys. 2016, 20 (2016)

[71] D. Glavan and C. Lin, Einstein-Gauss-Bonnet Gravity in Four-Dimensional Spacetime,Phys. Rev. Lett. 124, 081301 – 2020

[72] A. Abouelsaood, C.G. Callan, C.R. Nappi, S.A. Yost, Open strings in background gauge fields, Nuclear Physics B, Volume 280, 1987

[73] D. L. Wiltshire, Black holes in string-generated gravity models, Phys. Rev. D 38, 2445 –1988

[74] G.W. Gibbons, D.A. Rasheed, Electric-magnetic duality rotations in non-linear electrodynamics, Nuclear Physics B, Volume 454, Issues 1–2, 1995

[75] N. Bretón, Born-Infeld black hole in the isolated horizon framework, Phys. Rev. D 67, 124004 – 2003

[76] M. Aiello, R. Ferraro, and G. Giribet, Exact solutions of Lovelock-Born-Infeld black holes, Phys. Rev. D 70, 104014 – 2004

[77] D. N. Vollick, Born-Infeld-Einstein theory with matter, Phys. Rev. D 72, 084026 – 2005

[78] E. F. Eiroa, Gravitational lensing by Einstein-Born-Infeld black holes, Phys. Rev. D 73, 043002 – 2006

[79] D. A. Georgieva, I. Z. Stefanov, S. S. Yazadjiev, and M. D. Todorov, Born–Infeld black holes coupled to a massive scalar field, International Journal of Modern Physics D Vol. 20, No. 13, pp. 2471-2496 (2011)

[80] M. Allahverdizadeh, J. P. S. Lemos, and A. Sheykhi, Extremal Myers-Perry black holes coupled to Born-Infeld electrodynamics in five dimensions, Phys. Rev. D 87, 084002 – 2013

[81] T. K. Dey, Born–Infeld black holes in the presence of a cosmological constant,Physics Letters B, Volume 595, Issues 1–4, 2004

[82] Rong-Gen Cai, Da-Wei Pang, and A. Wang, Born-Infeld black holes in (A)dS spaces Phys. Rev. D 70, 124034 – 2004

[83] S. S. Yazadjiev, Einstein-Born-Infeld-dilaton black holes in nonasymptotically flat spacetimes Phys. Rev. D 72, 044006 – 2005

[84] M. H. Dehghani and H. R. Rastegar Sedehi, Thermodynamics of rotating black branes in (n+1)-dimensional Einstein-Born-Infeld gravity Phys. Rev. D 74, 124018 – 2006

[85] S. Fernando, Thermodynamics of Born-Infeld–anti-de Sitter black holes in the grand canonical ensemble, Phys. Rev. D 74, 104032 – Published 29 November 2006

[86] O. Mišković and R. Olea, Thermodynamics of Einstein-Born-Infeld black holes with negative cosmological constant, Phys. Rev. D 77, 124048 – 2008

[87] Y. S. Myung, Yong-Wan Kim, and Young-Jai Park, Thermodynamics and phase transitions in the Born-Infeld-anti-de Sitter black holes, Phys. Rev. D 78, 084002 – 2008

[88] Wei Yi-Huan, Energy and first law of thermodynamics for Born–Infeld–anti-de-Sitter black hole, Chinese Phys. B 19 090404 2010

[89] P. Chen, Thermodynamic geometry of the Born-Infeld-Anti-de-Sitter black holes, International Journal of Modern Physics A Vol. 26, No. 18, pp. 3091-3105 (2011)

[90] R. Banerjee and D. Roychowdhury, Critical behavior of Born-Infeld AdS black holes in higher dimensions, Phys. Rev. D 85, 104043 – 2012

[91] R. Banerjee and D. Roychowdhury, Critical phenomena in Born-Infeld AdS black holes, Phys. Rev. D 85, 044040 – 2012

[92] D. Kastor, S. Ray and J. Traschen, Classical and Quantum Gravity Enthalpy and the mechanics of AdS black holes, Class. Quantum Grav. 26 (2009) 195011

[93] Kubizňák, D., Mann, R.B., $P - V$ criticality of charged AdS black holes. J. High Energ. Phys. 2012, 33 (2012)

[94] Gunasekaran, S., Kubizňák, D. & Mann, R.B., Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization, J. High Energ. Phys. 2012, 110 (2012)

[95] B. P. Dolan, Where Is the PdV in the First Law of Black Hole Thermodynamics? Open Questions in Cosmology

[96] Shao-Wen Wei and Yu-Xiao Liu, Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes, Phys. Rev. D 87, 044014 – 2013

[97] B. P. Dolan, D. Kastor, D. Kubizňák, R. B. Mann, and J. Traschen, Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes, Phys. Rev. D 87, 104017 – 2013

[98] D. Kubizňák et al., Black hole chemistry: thermodynamics with Lambda, Class. Quantum Grav. 34 063001 2017

[99] Hansen, D., Kubizňák, D. & Mann, R.B. Universality of $P - V$ criticality in horizon thermodynamicsJ. High Energ. Phys. 2017, 47 (2017)

[100] Maldacena, J. The Large-N Limit of Superconformal Field Theories and Supergravity, International Journal of Theoretical Physics 38, 1113–1133 (1999)

[101] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from non-critical string theory, Physics Letters B, Volume 428, Issues 1–2, 1998

[102] Witten E., Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 253 1998

[103] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Large N field theories, string theory and gravity, Physics Reports, Volume 323, Issues 3–4, 2000

[104] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Building a Holographic Superconductor, Phys. Rev. Lett. 101, 031601 – 2008

[105] S. A. Hartnoll, Lectures on holographic methods for condensed matter physics, Class. Quantum Grav. 26 224002 2009

[106] A. Awad, S. R. Das, S. Nampuri, K. Narayan, and S. P. Trivedi, Gauge theories with time dependent couplings and their cosmological duals, Phys. Rev. D 79, 046004 – 2009

[107] S. R. Das, J. Michelson, K. Narayan, and S. P. Trivedi, Time-dependent cosmologies and their duals, Phys. Rev. D 74, 026002 – 2006

[108] S. R. Das, J. Michelson, K. Narayan, and S. P. Trivedi, Cosmologies with null singularities and their gauge theory duals, Phys. Rev. D 75, 026002 – 2007

[109] M. W. Zemansky and R. H. Dittman, Heat and Thermodynamics: An Intermediate Textbook (McGraw- Hill, New York, 1997)

[110] G. Ruppeiner, Riemannian geometry in thermodynamic fluctuation theory, Rev. Mod. Phys. 67, 605 – 1995

[111] A. Lala and D. Roychowdhury, Ehrenfest's scheme and thermodynamic geometry in Born-Infeld AdS black holes Phys. Rev. D 86, 084027 – 2012

[112] S. Guo and En-Wei Liang, Ehrenfest's scheme and microstructure for regular-AdS black hole in the extended phase space, Class. Quantum Grav. 38 125001 2021

[113] Jie-Xiong Mo and Wen-Biao Liu, Ehrenfest scheme for P−V criticality of higher dimensional charged black holes, rotating black holes, and Gauss-Bonnet AdS black holes, Phys. Rev. D 89, 084057 – 2014

[114] Mo, Jie-Xiong, Ehrenfest scheme for the extended phase space of f(R) black holes, EPL 105 20003, 2014

[115] G. Ruppeiner, Stability and fluctuations in black hole thermodynamics, Phys. Rev. D 75, 024037 – 2007

[116] G. Ruppeiner, Thermodynamic curvature and phase transitions in Kerr-Newman black holes, Phys. Rev. D 78, 024016 – 2008

[117] G. Ruppeiner, Thermodynamic curvature: pure fluids to black holes, J. Phys.: Conf. Ser. 410 012138 2013

[118] G. Ruppeiner, Stability and fluctuations in black hole thermodynamics, Phys. Rev. D 75, 024037 – 2007

[119] G. Ruppeiner, Thermodynamic Black Holes, Entropy 2018, 20(6), 460

[120] Åman, J.E., Bengtsson, I. & Pidokrajt, N. Geometry of Black Hole Thermodynamics General Relativity and Gravitation 35, 1733–1743 (2003)

[121] Bellucci, S., Tiwari, B.N. Thermodynamic geometry and Hawking radiation J. High Energ. Phys. 2010, 30 (2010)

[122] Liu, H., Lü, H., Luo, M. et al. Thermodynamical metrics and black hole phase transitions J. High Energ. Phys. 2010, 54 (2010)

[123] T. Sarkar et al. On the thermodynamic geometry of BTZ black holes, JHEP11(2006)015

[124] F. Weinhold, Metric geometry of equilibrium thermodynamics, J. Chem Phys. 63, 2479 (1975)

[125] D. Zou, Z. Yang, R. Yue and Peng Li, Thermodynamics of Gauss-Bonnet-Born-Infeld black holes in AdS space, Mod.Phys.Lett.A26:515-529,2011

[126] R. A. Hennigar, R. B. Mann, and E. Tjoa, Superfluid Black Holes, Phys. Rev. Lett. 118, 021301 (2017)

[127] N. Altamirano, D. Kubiznak, and R. B. Mann, Reentrant phase transitions in rotating anti–de Sitter black holes, Phys. Rev. D 88, 101502(R) (2013)

[128] B. P. Dolan, A. Kostouki, D. Kubiznak, and R. B. Mann, Isolated critical point from Lovelock gravity, Classical Quantum Gravity 31, 242001 (2014)

[129] C. V. Johnson, Holographic heat engines, Class. Quantum Grav. 31 205002 2014

[130] C. V. Johnson, Gauss–Bonnet black holes and holographic heat engines beyond large N, Class.Quant.Grav. 33 (2016) 21, 215009

[131] C. V. Johnson, Born-Infeld AdS Black Holes as Heat Engines, Class. Quantum Grav. 33 135001 2016

[132] A. Chakraborty, C. V. Johnson, Benchmarking Black Hole Heat Engines I, Int. J. Mod. Phys. D28, 1950012 (2019)

[133] A. Chakraborty, C. V. Johnson, Benchmarking Black Hole Heat Engines II Int. J. Mod. Phys. D28, 1950006 (2019)

[134] C. V. Johnson, F. Rosso, Holographic Heat Engines, Entanglement Entropy, and Renormalization Group Flow, Class.Quant.Grav. 36 (2019) 015019

[135] C. V. Johnson, Holographic Heat Engines as Quantum Heat Engines, Class. Quant. Grav. 37, 034001 (2019)

[136] C. V. Johnson, de Sitter Black Holes, Schottky Peaks, and Continuous Heat Engines, arXiv:1907.05883 [hep-th]

[137] C. V. Johnson, Exact model of the power-to-efficiency trade-off while approaching the Carnot limit, Phys. Rev. D 98, 026008 – 2018

[138] B. P. Dolan, Bose condensation and branes, J. High Energy Phys. 10 (2014) 179.

[139] D. Kastor, S. Ray and J. Traschen, Chemical potential in the first law for holographic entanglement entropy, J. High Energ. Phys. 2014 (2014) 120.

[140] J.-L. Zhang, R.-G. Cai and H. Yu, Phase transition and thermodynamical geometry for Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime, J. High Energy Phys. 2015 (2015) 143.

[141] J.-L. Zhang, R.-G. Cai and H. Yu, Phase transition and thermodynamical geometry of Reissner-Nordstrom-AdS black holes in extended phase space, Phys. Rev. D 91 (2015) 044028.

[142] B. P. Dolan, Pressure and Compressibility of Conformal Field Theories from the AdS/CFT Correspondence Entropy 18 (2016) 169.

[143] F. McCarthy, D. Kubizňák and R. B. Mann, Breakdown of the equal area law for holographic entanglement entropy, J. High Energy Phys. 2017 (2017) 165.

[144] C. V. Johnson, V. L. Martin and A. Svesko, Microscopic description of thermodynamic volume in extended black hole thermodynamics, Phys. Rev. D 101, 086006(2020)

[145] A. Karch and B. Robinson, Holographic black hole chemistry, J. High Energ. Phys. 2015, 1–15

[146] W. Cong, D. Kubizňák, and R. B. Mann, Thermodynamics of AdS Black Holes: Critical Behavior of the Central Charge, Phys. Rev. Lett. 127, 091301 – 2021